THE

MESSENGER OF MATHEMATICS,

EDITED BY

W. ALLEN WHITWORTH, M.A.,
FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

C. TAYLOR, M.A.,
FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

W. J. LEWIS, B.A.,
FELLOW OF ORIEL COLLEGE, OXFORD.

R. PENDLEBURY, B.A.,
FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

J. W. L. GLAISHER, B.A., F.R.A.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. I.

MACMILLAN AND CO.,
London and Cambridge.
EDINBURGH: EDMONSTON & DOUGLAS. GLASGOW: JAMES MACLEHOSE.
DUBLIN: HODGES, FOSTER & CO. OXFORD: JOHN HENRY AND J. PARKER.
1872.

CAMBRIDGE:

PRINTED BY W. METCALFE AND SONS, GREEN STREET.

ADVERTISEMENT.

In announcing the commencement of a new series, the Editors desire to explain the modifications which will distinguish it from the former series.

The *Messenger of Mathematics* was projected about ten years ago, chiefly with the view of encouraging original research in the three Universities, among junior graduates and others. It was thought that through the *Messenger* many valuable papers might be made public which their authors would not have deemed of sufficient interest to communicate to Scientific Societies. An examination of the Five Volumes already published will make it evident that the Editors have throughout endeavoured to keep their original purpose steadily in view. While feeling, however, that they have every reason to be satisfied with the success achieved by the *Messenger* regarded as a stimulus to original research in junior students, they have also great satisfaction in acknowledging that no inconsiderable proportion of its contents have been supplied by writers of established reputation, who rank amongst the foremost mathematicians of the age; and it is this fact in particular which now induces them to appeal directly to the mathematical world at large, and to remove from their title-page any words which might be supposed to limit the sphere of usefulness of the *Messenger*.

In the Universities a marked progress has been lately witnessed. Their whole mathematical system is being largely affected by the spirit of the age; the range of subjects is increased; teaching is being more and more specialized; and it is anticipated that lecturers and students, enabled at length to concentrate their energies in a way heretofore impracticable, will manifest an increased literary activity, and produce a more continuous and ample supply of those original investigations, which it is the object of the *Messenger* to foster. The Editors however, as above intimated, are very far from desiring to restrict the *Messenger* to contributions received from resident members of the three Universities; they are prepared to receive communications from every available source, and have most gladly welcomed articles which have reached them from such distant centres as Queensland, Italy, and the United States.

Besides original papers, it is intended to insert brief notices of select articles or treatises on mathematical subjects, as well as short accounts of the proceedings of societies at home or abroad which may be communicated.

It is intended that each volume of the *Messenger* shall consist of not less than twelve sheets, to be published within one year. The Editors will endeavour, as far as possible, to publish one number every month, without restricting themselves to any particular day; but even this amount of uniformity is not always possible in mathematical journals, and the appearance of twelve numbers in the course of the year is all that the Editors feel able to promise with certainty.

Cambridge,
March, 1871.

PREFACE.

The Editors take the opportunity, afforded by the completion of the First Volume of the New Series, to express their satisfaction at the success that has attended the *Messenger of Mathematics* during the past year. A Monthly Journal wholly devoted to Mathematical Science is, at all events in this country, a novelty, and on this account the reception of the *Messenger* is all the more gratifying. The hope expressed in the advertisement announcing the New Series, with regard to the supply of original investigations, has been most amply justified; in fact, the only difficulty experienced has been one of selection.

The advantages derived from the appearance of a Journal at brief intervals in the matter of quick publication of papers are obvious; but it should be remarked that, in consequence of the necessarily somewhat limited size of the numbers,

a paper of considerable length must sometimes either be divided or delayed till a more favourable opportunity may present itself, in order to afford variety to each number. It is, however, almost needless to add that the inconvenience arising from this cause is very slight, when compared with the substantial advantage caused by a general freedom from delay.

No alteration will be made in the *Messenger of Mathematics*, which will appear during the next year in the same form as it has during that just completed.

Cambridge,
April, 1872.

ERRATA.

Page 15, line 5, *for* investigation, *read* instigation.
" 25, " 22, *for* recent sines, *read* recent times.

CONTENTS.

PURE GEOMETRY.

ANALYTICAL GEOMETRY OF TWO AND THREE DIMENSIONS.

ALGEBRA AND TRIGONOMETRY.

DIFFERENTIAL AND INTEGRAL CALCULUS.

MECHANICS, HYDROSTATICS, LIGHT, VIBRATIONS OF RODS, &c.

MISCELLANEOUS.

TRANSACTIONS OF SOCIETIES.

REVIEWS.

MESSENGER OF MATHEMATICS.

EXPLANATION OF A DYNAMICAL PARADOX.

By Professor *G. G. Stokes.*

THE answer to the following question, proposed in the Smith's Prize Examination, 1871, is sent in compliance with a request from one of the Editors:

"*In a compound pendulum consisting of masses m, m' attached to strings of length l, l', in which of course the most general small motion in one plane consists of two harmonic vibrations superposed, if the upper mass m be very large compared with the under mass m', it is clear that one of the two periodic times (that corresponding to the mode of vibration in which m is nearly at rest) must be very nearly the same as in a simple pendulum of length l', and the other very nearly the same as in a simple pendulum of length l. By a continuous variation of l', the former may be made to pass continuously from less to greater than the latter, and therefore for some value of l' nearly equal to l the two must be equal. But when a system is in stable equilibrium (as is clearly the case here) the equation the roots of which give the times of vibration cannot have equal roots, for that would imply the transitional condition between stable and unstable.*

"*Point out precisely the fallacy which leads to the above contradiction.*"

The fallacy lies in the tacit assumption that it is *the same* root of the quadratic which determines the times of vibration that correspond throughout to the same approximate physical state, *i.e.* the state in which (not considering the special case in which l, l' are nearly equal) the upper mass is nearly at rest, or the two masses move through comparable spaces, as the case may be. Let T, T' be the two

times of vibration (or half periods) τ, τ' the times of vibration of simple pendulums of lengths l, l'; and suppose m, m', l, l' to change continuously, yet so that l always remains distinctly greater than or distinctly less than l'; *i.e.* so that the ratio of $l \sim l'$ to l or l', though absolutely, it may be, small, remains finite while $m' : m$ may be taken as small as we please. Of the two T, T', let T be that which for one set of values of the constants m, m', l, l' is nearly equal to τ; then must the same root T remain throughout that which is nearly equal to τ; for it is obliged to be nearly equal to one of the two τ, τ' which do not become nearly equal to each other. But if we suppose l, l' to change continuously, so that l, from having been distinctly less, becomes distinctly greater than l', and if T be that root which for $l < l'$ is nearly equal to τ, since τ, τ' pass through equality, and T is merely known to be nearly equal to one of them, there is nothing to shew which of the two τ, τ' it is that T is nearly equal to when $l > l'$. By the general principle referred to in the second part of the question, we see that it must be τ'.

The same thing may of course be shewn by the direct solution of the problem. Putting n for $\frac{\pi}{T}$, we find by the usual methods

$$mll'n^4 - (m+m')(l+l')gn^2 + (m+m')g^2 = 0,$$

g being gravity, and the roots of this quadratic in n^2 are

$$n^2 = \frac{m+m'}{2m} g \left[\frac{1}{l} + \frac{1}{l'} \pm \sqrt{\left\{\left(\frac{1}{l} + \frac{1}{l'}\right)^2 - \frac{m}{m+m'}\frac{4}{ll'}\right\}}\right].$$

If m' be very small, and l, l' not very nearly equal, the radical becomes very nearly $\frac{1}{l} \sim \frac{1}{l'}$; and denoting by n^2, n'^2 the roots corresponding to the signs $+$, $-$, respectively, we have very nearly

$$n^2 = \frac{g}{2}\left\{\frac{1}{l} + \frac{1}{l'} + \left(\frac{1}{l} \sim \frac{1}{l'}\right)\right\}, \quad n'^2 = \frac{g}{2}\left\{\frac{1}{l} + \frac{1}{l'} - \left(\frac{1}{l} \sim \frac{1}{l'}\right)\right\}.$$

If $l < l'$, we have

$$n^2 = \frac{g}{l}, \quad n'^2 = \frac{g}{l'};$$

but if $l > l'$,

$$n^2 = \frac{g}{l'}, \quad n'^2 = \frac{g}{l}.$$

When l, l' are nearly equal, we can no longer distinguish the two harmonic vibrations by the character, that from one of them m is nearly at rest, while the small mass m' moves considerably, since the motion of m as compared with m' is comparable in the two. In fact, the harmonic vibrations, which, when l, l' are distinctly different, are characterized by the properties above referred to, have their properties interchanged when $l : l'$ passes through 1.

NOTE ON THE PROBLEM OF ENVELOPES.

By Professor *Cayley*.

THERE is a mode of looking at the problem of Envelopes, which, so far as I am aware, has not been explicitly noticed. Let $U=(x, y, z)^m$ be a function of the coordinates (x, y, z), $\Theta=\Theta'=(x, y, z)^a (x', y', z')^{a'}$ a function of the two sets of coordinates (x, y, z) and (x', y', z'); it being understood that when we write Θ we regard (x, y, z) as the current coordinates, when Θ' we regard (x', y', z') as the current coordinates. Suppose that we have $U=0$; the curve $\Theta'=0$ is then a curve the equation whereof contains as parameters the coordinates (x, y, z) of a point P on the curve $U=0$; and we may seek for the envelope of the curve $\Theta'=0$ as P describes the curve $U=0$; the required envelope is of course obtained as an equation in (x', y', z') given by the elimination of x, y, z, λ from the equations (equivalent to four equations only)

$$U=0,\ \Theta'=0,$$
$$d_x\Theta'+\lambda d_x U=0,$$
$$d_y\Theta'+\lambda d_y U=0,$$
$$d_z\Theta'+\lambda d_z U=0.$$

But, observe that the required envelope is the locus of the points of intersection of the curve $\Theta'=0$ belonging to a particular point (x, y, z) of the curve $U=0$, by the curve $\Theta'=0$ which belongs to a consecutive point of U. The curve $\Theta=0$, considering therein (x', y', z') as the coordinates of a given point of the plane, determines by its intersection with $U=0$ those points (x, y, z) on the curve $U=0$, to each of which belongs a curve $\Theta'=0$ passing through the point in question (x', y', z'). Hence, if the curve $\Theta=0$ touch the curve $U=0$, the point of contact, coordinates (x, y, z), is

a point such that to it and to the consecutive point there belong curves, each of them passing through the given point (x', y', z'). Hence expressing that the curves $\Theta=0$, $U=0$ touch each other, we have a relation in (x', y', z') which is the locus of the point of intersection of the curves $\Theta'=0$ which belong to two consecutive points of the curve $U=0$; that is, the equation of the required envelope is obtained as the condition that the curves $U=0$, $\Theta=0$ shall touch each other. But when the curves touch each other, they have at the point of contact their derived functions proportional, or we have simultaneously

$$\begin{aligned}
U &= 0, \quad \Theta = 0, \\
d_x\Theta + \lambda d_x U &= 0, \\
d_y\Theta + \lambda d_y U &= 0, \\
d_z\Theta + \lambda d_z U &= 0,
\end{aligned}$$

the same equations as before, since Θ and Θ' denote the same function.

It is to be added that when $a=m$, the equations

$$\begin{aligned}
d_x\Theta + \lambda d_x U &= 0, \\
d_y\Theta + \lambda d_y U &= 0, \\
d_z\Theta + \lambda d_z U &= 0,
\end{aligned}$$

are homogeneous in (x, y, z), and we may by the elimination of (x, y, z) from these equations obtain an equation Disct. $(\Theta+\lambda U)=0$, say for shortness $\Lambda=0$, involving λ and also the coordinates (x', y', z'). Now it is a known theorem that the condition for the contact of the two curves $U=0$, $\Theta=0$ can be obtained by expressing that the equation $\Lambda=0$ shall have a pair of equal roots, or what is the same thing, by equating to zero the discriminant of the function Λ; this last-mentioned process leads therefore to the equation of the envelope of the curve $\Theta'=0$, viz. (a being $=m$ as above) the equation of the envelope of the curve $\Theta'=0$, is in fact

$$\text{Disct.}_\lambda \ \text{Disct.}_{(x, y, z)} (\Theta+\lambda U)=0,$$

viz. we first take the discriminant of the function $\Theta+\lambda U$ in regard to the coordinates (x, y, z), and then taking the discriminant in regard to λ of this discriminant we equate it to zero. This is in many cases a more simple process than that of the direct elimination of x, y, z, λ from the five equations.

EXERCISES IN THE INTEGRAL CALCULUS.

By *Sir James Cockle, F.R.S.*

THE present paper will consist of two sections. The first will be introductory. In the second I shall treat of certain leading equations and their transformations. When citing Boole, I refer to the second (posthumous) edition of his *Differential Equations* (1865), and the Supplementary Volume (1865) thereto. These papers will be restricted to the subject of linear differential equations of the second order, or, using a term of Mr. De Morgan's,* linear biordinals. The subject of linear equations is of primary importance (Boole, p. 192). Equations whose symbolical form is binomial, generally admit of solution by definite integrals (Boole, p. 477).

§ 1. *Introductory.*

1. The term quantic is, or is likely to be, recognized as meaning a rational, entire, and homogeneous function of variables. Thus, $ax + by$ is a quantic. Mr. De Morgan says† that quotic is what was meant. I do not presume to offer an opinion on the subject, but, in order to avoid circumlocution and for a special purpose, I shall adopt the word quotic, and use it as meaning a rational and entire, but not generally homogeneous function. Thus, I call $ax + b$ a quotic in x. If by a root of an expression, we mean a value of the dependent, in terms of the independent, variable, which, substituted in the expression, causes it to vanish, then it is correct, and often will be convenient, to speak of the root of an expression, rather than of a corresponding equation. And Mr. De Morgan has so employed the term root. Thus we may say that a is the root of the expression $x - a$, as well as that a is the root of the equation $x - a = 0$.

---

* On some points in the Theory of Differential Equations, *Camb. Trans.* Vol. IX., Part IV.

† On the Root of any Function; and on Neutral Series, No. II. *Camb. Trans.*, Vol. XI., Part II.

2. When to any operand, say z, any symbolical quotic, say $f\left(\frac{d}{d\xi}\right)$, in any differential symbol $\frac{d}{d\xi}$, is prefixed as an operator, I shall, for the sake of brevity, call the result, $f\left(\frac{d}{d\xi}\right)z$, a quotoid. Thus, $M\frac{dz}{d\xi}+Nz$ is a quotoid. The corresponding homogeneous forms, such as $Mdz+Nzd\xi$, I shall call quantoids. When a symbolical quotic is capable of decomposition into symbolical factors, whereof each is a symbolical quotic, I call the decomposition a symbolical decomposition.

3. Boole (Suppl. p. 190) regarded as primary those forms of binomial equations which are integrable, but not through any reduction effected by his general method. He (pp. 428–429) recognized two allied forms as primary, and he appears (Suppl. p. 190) to have subsequently inclined to the opinion, that all primary forms are differential resolvents of algebraical equations. However this be, it is not difficult to show why Boole's method fails to solve the allied primary forms. We have seen (*ante*, Vol. III., pp. 49–50) that the equation

$$\frac{d^2y}{dx^2}-\left\{\frac{1}{2r}\frac{dr}{dx}+a\sqrt{r}\right\}\frac{dy}{dx}+ry=0,$$

is soluble by changing the independent variable. The symbolical decomposition of this equation is

$$\left\{\frac{d}{dx}-\frac{1}{2r}\frac{dr}{dx}+\beta\sqrt{r}\right\}\left\{\frac{d}{dx}+\gamma\sqrt{r}\right\}y=0,$$

wherein β and γ are roots of the quadratic

$$\beta^2-a\beta+1=0.$$

This decomposition admits of transformation (*ante*, Vol. I., pp. 167–168, foot-note). Take either of the symbolical factors, say the latter, and transform‡ it. We cannot in

---

* The process of the latter transformation in the footnote at pp. 167–168 of Vol. I. should be amended as follows:

$$\lambda Y=(U_1)(\lambda U_2)-\frac{d\lambda}{dx}\frac{d}{dx}-u\frac{d\lambda}{dx}$$

$$=(U_1)(\lambda U_2)-\frac{d\lambda}{dx}(U_2-u)-u\frac{d\lambda}{dx}$$

$$=\left(U_1-\frac{1}{\lambda}\frac{d\lambda}{dx}\right)(\lambda U_2),$$

general get rid of the radical. But the absence of radicals is essential to the operation of Boole's method, which proceeds by way of rational symbolical decomposition. If $\sqrt{r}$ is not an irreducible surd the difficulty will not arise. If it is, then, on the assumption that the equation has been subjected to every reduction necessary to prepare it for the application of Boole's process, we must suppose that $a=0$ and $\beta=-\gamma=\pm\sqrt{(-1)}$. We must further suppose that r is a rational fraction, and consequently that $\sqrt{r}$ is capable of being written in the form $Q\sqrt{R}$, where Q is rational, and R is rational and entire, and $\sqrt{R}$ is an irreducible quadratic surd. Moreover, we may consider that R is a binomial or polynomial expression, for if R be a monomial, a change of the independent variable will expel, or will have expelled,

---

which may be identified with the first transformation. Again, we have

$$\lambda Y=(U_1)(\lambda U_2)-\frac{d\lambda}{dx}(U_1-U)-u\frac{d\lambda}{dx}$$

$$=(U_1)(\lambda U_2)-\frac{d\lambda}{dx}U_1+\frac{d\lambda}{dx}(U-u)$$

$$=(U_1)(\lambda U_2)-(U_1)\frac{d\lambda}{dx}+\frac{d^2\lambda}{dx^2}+\frac{d\lambda}{dx}(U-u)$$

$$=(U_1)\left(\lambda U_2-\frac{d\lambda}{dx}\right)+\frac{d^2\lambda}{dx^2}+\frac{d\lambda}{dx}(U-u).$$

Hence if $$\frac{d^2\lambda}{dx^2}+\frac{d\lambda}{dx}(U-u)=0 \quad\ldots\ldots\ldots\ldots\ldots (A),$$

we have the transformed decomposition

$$(U_1)\left(\lambda U_2-\frac{d\lambda}{dx}\right)=0.$$

In the last line of the footnote to page 167 (Vol. I,) and throughout the rest of the footnote, with the exception of the final symbolical decomposition, $+u$ should replace $-u$. The final result is correct. By combining the two transformations, we obtain the double transformation

$$\mu\lambda Y=\left(\frac{d}{dx}+U-\frac{1}{\mu}\frac{d\mu}{dx}\right)\left(\mu\lambda\frac{d}{dx}+\mu\lambda u-\mu\frac{d\lambda}{dx}\right),$$

wherein μ is arbitrary, and λ is determined by (A). This result may be put under the form

$$\mu\lambda Y=\left(U_1-\frac{1}{\mu}\frac{d\mu}{dx}\right)\mu\lambda\left(U_2-\frac{1}{\lambda}\frac{d\lambda}{dx}\right),$$

where the sinister factor on the dexter side is supposed to operate on all that follows it. The stellar notation of Professor Sylvester would probably be useful in researches on symbolical decomposition. My results at pp. 43–44 of Vol. III. of the *Messenger* may be generalized by saying, that the finding of an external factor of a quotoid, and the finding of an integrating factor thereof, are nearly the same question. For, $e^{\int U dx}$ is an integrating factor of

$$\left(\frac{d}{dx}+U\right)Q,$$

whatever Q may be.

the radicality. Now if R is a binomial, or a polynomial, Boole's transformations will not eliminate the irreducible surd $\sqrt{R}$. The change of x into ax, of y into $x^n y$, or of x into $\frac{1}{x}$ (compare Suppl. pp. 185–186), will not effect this elimination, nor will Prop. III. (p. 419).

4. Such, or some such, is, I think, the general explanation of the primary forms. There are, however, particular cases which merit notice, and, as the subject of primary forms is one of interest and importance, I hope to recur to their consideration with greater detail.

5. I have already, in the *Messenger*, adverted to certain functions which I called characteristic (Vol. I., p. 241, Art. 46) or critical (Vol. III., p. 47, Art. 76). Their theory is susceptible of an indefinite extension which, as it relates to equations of the higher orders, I shall not enter* upon here. But inasmuch as these functions, which I now call criticoids, are useful in the treatment of linear biordinals, and have in fact led me to some of the results given in the next section, I shall deduce the criticoid of the biordinal by a process capable of extension to equations of any order.

6. Suppose that in ordinary algebra we have given the relations

$$f(x+a_1) = x^2 + 2a_1x + a_2 \dots\dots\dots\dots (a),$$

$$f(x+A_1) = x^2 + 2A_1x + A_2 \dots\dots\dots\dots (A).$$

Then from (a) we deduce

$$\begin{aligned} f(x) &= (x-a_1)^2 + 2a_1(x-a_1) + a_2 \\ &= x^2 + a_2 - a_1^2 \dots\dots\dots\dots\dots\dots (b). \end{aligned}$$

So from (A) we obtain

$$f(x) = A_2 - A_1^2 \dots\dots\dots\dots\dots\dots (B),$$

and, equating these values of $f(x)$ we have, on reduction,

$$a_2 - a_1^2 = A_2 - A_1^2 \dots\dots\dots\dots (1).$$

In other words, either side of (1) is a critical function, and unchanged when $f(x)$ becomes $f(x+\lambda)$, where λ is indeterminate.

* The reader will find an outline of the higher theory in my paper "On Criticoids," in the *Philosophical Magazine*, (Ser. 4, Vol. XXXIX., p. 201), for March, 1870.

7. In like manner let

$$f\left(e^{\int a_1 dx} y\right) = e^{\int a_1 dx}\left(\frac{d^2y}{dx^2} + 2a_1\frac{dy}{dx} + a_2\right) \text{.........} (2),$$

$$f\left(e^{\int A_1 dx} y\right) = e^{\int A_1 dx}\left(\frac{d^2y}{dx^2} + 2A_1\frac{dy}{dx} + A_2\right) \text{.......} (3).$$

Then, since

$$\frac{d}{dx}\left(e^{-\int a_1 dx} y\right) = e^{-\int a_1 dx}\left(\frac{dy}{dx} - a_1 y\right),$$

and

$$\frac{d^2}{dx^2}\left(e^{-\int a_1 dx} y\right) = e^{-\int a_1 dx}\left\{\frac{d^2y}{dx^2} - 2a_1\frac{dy}{dx} + \left(a_1^2 - \frac{da_1}{dx}\right)y\right\},$$

from (2) we deduce

$$f(y) = \frac{d^2y}{dx^2} + \left(a_2 - a_1^2 - \frac{da_1}{dx}\right)y \text{............} (4).$$

So from (3) we obtain

$$f(y) = \frac{d^2y}{dx^2} + \left(A_2 - A_1^2 - \frac{dA_1}{dx}\right)y \text{...........} (5),$$

and, equating these values of $f(y)$, we find

$$a_2 - a_1^2 - \frac{da_1}{dx} = A_2 - A_1^2 - \frac{dA_1}{dx} \text{............} (6).$$

In other words, either side of (6) is a criticoid, and unchanged when $f(y)$ becomes $f(\lambda y)$, where λ is indeterminate and $f(\lambda y)$ is, of course, written in the form $\lambda\phi(y)$.

8. The algebraical and the differential results have a common origin, which may be the source of a still wider generalization. Let ϕ be any function or operation, such that

$$\phi(a+b) = \phi(a) + \phi(b) \text{.................} (7).$$

Also let

$$F_2(u) = u_2 - u_1^2 - \phi(u_1) \text{.................} (8),$$

and let the six symbols u_1, u_2, a_1, a_2, A_1 and A_2 be connected by the two relations

$$u_1 + a_1 = A_1 \text{........................} (9),$$

$$u_2 + 2a_1u_1 + a_2 = A_2 \text{.................} (10),$$

then we have identically

$$F_2(u) + F_2(a) = F_2(A) \text{...............} (11).$$

Now equations (9) and (10) occur in the linear transformation of quotics, and the factorial transformation of quotoids. In the former case $u_2 = u_1^2$ and $\phi(u) = 0 = \phi(a) = \phi(A)$.

In the latter case

$$u_2 = u_1^2 + \frac{du_1}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

so that, when $F_2(u)$ vanishes, $\phi(u_1)$ is $\frac{du_1}{dx}$. The evanescence of $F_2(u)$ leads to the critical or criticoidal equation $F_2(a) = F_2(A)$. The process of this article admits of generalization.

9. The equations which I am about to discuss, and which I term leading equations, occupy, or at all events they and their transformations occupy, a very considerable portion of the field of solved biordinals. In the theory of differential equations, and independently of Boole's processes, there are propositions which seem to deserve the name of general. For instance, I have* shown that the solution of

$$\frac{d^2y}{dx^2} + \phi\left(\pm\frac{1}{x}\right)\frac{1}{x^4}\,y = 0,$$

may be made to depend upon that of

$$\frac{d^2y}{dx^2} + \phi(x)\,y = 0,$$

where $\phi(x)$ is any function whatever of x. The leading equations possess general properties, though the generality is of a nature different to that found in the instance cited.

10. Criticoids are analogous to critical functions or seminvariants in algebra, and the analogies extend to linear differential equations of all orders. In the mode of deducing the criticoid given in a preceding volume (*ante*, Vol. I., p. 121, Art. 11), the existence of the analogy is not sufficiently noticed. In Arts. 6 and 7 of this paper I have to some extent remedied the defect. The algebraical proposition in Art. 6 might perhaps be more simply proved, but it is demonstrated in that particular form in order to avoid assuming that a quadric or quadratic has a finite solution. If, by means of analogy, we attempted to extend to linear biordinals, or higher ordinals, any proposition deduced by

* See the Reprint of the *Educational Times*, Vol. VI., p. 33, Quest. 1905. See also *Ibid*, Vol. V., p. 50, Quest. 1854, and Vol. VII., p. 70, Quest. 1889, where I have discussed transformations which may have a bearing upon the subject of leading equations.

taking for granted the existence of roots of algebraical equations, the analogy would fail us. By deducing the algebraical result without making the assumption we obtain the analogue of a critical function, viz. a criticoid. It should be observed that if the coefficients of the quotics

$$x^2 + 2a_1 x + a_2, \quad X^2 + 2A_1 X + A_2$$

are connected by the relation (1), then the roots of those quotics are connected by the relation

$$x + a_1 = X + A_1 \text{ .... } (13),$$

and if the coefficients of the quotoids

$$\frac{d^2y}{dx^2} + 2a_1 \frac{dy}{dx} + a_2 y$$

$$\frac{d^2Y}{dx^2} + 2A_1 \frac{dY}{dx} + A_2 Y$$

are connected by the relation (6), then the roots of the respective quotoids are connected by the equation

$$e^{\int a_1 dx} y = e^{\int A_1 dx} Y \text{ .... } (14),$$

wherein the sign of integration may involve* an arbitrary constant.

11. Application of the method of criticoids has satisfied me of its practical utility. By its aid we may, often by mere inspection, ascertain the proper mode of dealing with a proposed quotoid. Where the coefficients of a quotoid are variable a change of the independent variable may, in general, so disguise an equation that, apart from our original knowledge of its form, we could not retrace our steps to it. But when, by a factorial substitution, one linear biordinal is transformed into another, the criticoid enables us to trace the transformation and the transformed quotoid is stripped of its disguise. The process of Boole does not supersede or enable us to dispense with a consideration of criticoids. The latter functions may avail us when we are seeking to transform a given quotoid into another with rational quotics for coefficients. Where the process of Boole does apply, a solu-

---

* Some remarks on the necessity of the introduction of arbitrary functions to restore the required identity of the expressions deduced for the same differential coefficients will be found in a footnote at pp. 215–216 of Peacock's Report (to the Third Meeting of the British Association). So far as differentiation is concerned $\frac{ax+b}{cx+e}$ and $\frac{f}{cx+e}$ are substantially the same form.

tion may sometimes be obtained with equal, or even greater, ease by criticoids. The use of criticoids will be facilitated by the following notation. Let q and Q be biordinal quotoids, and let

$$q \equiv Q$$

mean, not the identity of q and Q, but that q and Q have the same criticoid. Then it would be a useless labour to write out in detail all the transformations, for if we have a final result such as

$$q \equiv Q \equiv Q_2$$

all that we are concerned to know is, in general, the connection between the roots of q and Q_2, and that we can determine without ascertaining the roots of Q, or their relation to the roots of either of the other quotoids.

12. I shall now give an example (from Boole, p. 403) in which it seems to me that the criticoidal process has the advantage. We see at a glance, and without any calculation, that it is necessary to write down that

$$\frac{d^2y}{dx^2} - (2x+1)\frac{dy}{dx} + (x^2+x-1)\,y \equiv \frac{d^2Y}{dx^2} - \tfrac{1}{4}Y,$$

and that the coefficients of the latter quotoid are zero or constant, and further that

$$e^{-\frac{x^2+x}{2}}y = CY = C_1 e^{\frac{x}{2}} + C_{-1}e^{-\frac{x}{2}},$$

or

$$y = c_1 e^{\frac{(x+1)^2}{2}} - c_2 e^{\frac{x^2}{2}},$$

which is equivalent to Boole's result. The criticoidal process also applies immediately to two other general examples of Boole (p. 411, Ex. 11, and pp. 459–460, Ex. 14), to an interesting example (Ex. 12 of p. 432) connected with the solution of a physical problem, as well as to other* examples to be found in his work. I hope on another occasion to exemplify the process more fully.

* The criticoid process yields a ready solution of the following examples in Boole, viz. Ex. (2) of p. 20; Ex. 11 of p. 207; and by a double transformation it reduces Ex. 9 and Ex. 10 of p. 207, and, by a single transformation, Ex. 5 of p. 234, to forms immediately integrable. It also reduces Ex. 14 of p. 208, Ex. 6 of p. 422, and Ex. 8 of p. 424, to leading equations, and it gives a corresponding modification of form to Ex. 9 of p. 425. It also applies to the equation of Laplace's functions, either independently or as an easy verification of Boole's process in Ex. 13 of pp. 433–434.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

At a Meeting of this Society, held Thursday, April 13th, at 8 P.M., W. Spottiswoode, Esq., F.R.S., President, in the Chair, the following gentlemen were elected Members: the Hon. J. W. Strutt, Trinity College, Cambridge; Major F. Close, M.A.; Mr. James Stuart, B.A., Fellow and Assistant Tutor, Trinity College, Cambridge. Mr. C. J. Monro, B.A., late Fellow of the same College was proposed for election.

Before proceeding to the reading of papers, the President briefly alluded to the great loss the Society had sustained in consequence of the death of its first President, the late Prof. De Morgan. Dr. Hirst, who had been well acquainted with Mr. De Morgan for some years, having been associated with him as a Professor in University College, gave a sketch of the work done by the deceased, especially dwelling upon the originality and acuteness displayed in all his writings, pointing out the fact that in his Double Algebra was perhaps to be found the first trace of what was subsequently expanded by Sir W. R. Hamilton into the Theory of Quaternions; in the treatise on the Differential and Integral Calculus were to be found the germs of much of what has subsequently come prominently forth under the hands of Boole and others, and dwelling on the ability displayed in the Formal Logic. He then mentioned how great an interest Mr. De Morgan had taken in the Society since its very commencement, which led him to undertake the office of President and to deliver the inaugural address.

Dr. Hirst then, after stating that it was Mr. De Morgan who further did away with the original restriction of Membership to persons associated with University College, paid a cordial and feeling tribute to the personal character of the deceased.

Prof. Crofton, F.R.S., then explained his Diagrams illustrative of the "Stresses in Warren and Lattice Girders." He stated that he had not found anything to help him in English Text-books, and referring to papers by Rankine and Clerk Maxwell. Prof. Henrici and Mr. Merrifield in their remarks on the communication, drew attention to a work little known in this country, by Culmann (Die Graphische Statik, 1866), in which Mr. Crofton's constructions had been anticipated, and the methods applied to a very wide range of subjects; Prof. Henrici further illustrated the subject of Prof. Crofton's remarks by a very simple and ingenious construction.

Prof. Cayley, V.P., followed with a brief sketch of the contents of his third Memoir on "Quartic Surfaces."

R. Tucker, M.A., *Hon. Sec.*

Proceedings of the Association for the Improvement of Geometrical Teaching.

A letter written to *Nature* by Mr. Rawdon Levett, about a year ago, led to the formation of an Association having for its immediate objects:

1. To collect and distribute information as to the prevailing methods of instruction in Geometry practised in this and other countries, and to ascertain whether the desire for change was general.

2. To use its influence to induce examining bodies to frame their questions in Geometry without reference to any particular text-book.

3. To stamp with its approval some text-book already published, or to bring out a new one under its own auspices.

The first annual report—dated Jan., 1871—contains an inaugural address by Dr. Hirst, who was unanimously called to the chair.

Dr. Hirst is unsparing in his condemnation of Euclid—"I know no successful teacher who will not admit that his success is almost in proportion to the liberty he gives himself to depart from the strict line of Euclid's Elements, and to give the subject a life which without that departure it could not possess. I know no geometer who has read Euclid *critically;* no teacher, who has paid attention to modes of exposition, who does not admit that Euclid's Elements are full of defects. They 'swarm with faults' in fact, as was said by an eminent Professor of this College,* who has helped to train, perhaps, some of the most vigorous mathematical thinkers of our day."

* University College, London.

But as the President goes on to warn the Association, the task before them can be no light one, when, in France, some of the most competent authorities, as Duhamel and Houël, complain as bitterly of the state of Geometrical teaching there as we do of ours here, or express their extreme regret at the disuse into which Euclid's Elements have fallen; and when Italy, at the investigation of Professors Cremona and Battaglini, has thought itself compelled to take "a retrograde step," and to readopt Euclid *pure* and *simple*, though only in classical schools.

"Seeing then that numerous attempted emendations of Euclid have not met with the approval of the Geometrical world at large, but have on the contrary caused reactionary steps to be taken, would it not be one of the most useful functions of this Association to endeavour by its influence—its severe and conscientious criticism if needful—to discourage the too hasty publication of Geometrical treatises in England, to insist on the subordination of mere individual predilections, and to represent the dangers attending the treatment of fundamental and really important parts of the subject, without that long and careful consideration which centuries have proved to be needful."

It was subsequently resolved, *inter alia*, that all the members of the Association be invited to furnish the Committee of Management, by the first of May, 1871,* with programmes, or suggestions relative to a programme, of the subjects which a text-book on Geometry ought to include; and that the Committee be requested to report at the next general meeting of the Association—viz. on or about Jan. 17, 1872—upon all the programmes and suggestions received from the members.

REVIEW.

Elementary Geometry. Part I. Containing the first two books of Euclid with Exercises and Notes. By J. Hamblin Smith, M.A.

"To preserve Euclid's order, to supply omissions, to remove defects, to give brief notes of explanation, and simpler methods of proof in cases of acknowledged difficulty—such are the main objects of this Edition of the First and Second Books of the Elements......... In the Third Book, which I am now preparing, I intend to deviate with even greater boldness from the precise line of Euclid's methods."

In this first part the work is well done so far as it goes; but when an improved demonstration of I. 5 has been given, why should Euclid's proof still be retained in the text? and when the case of two triangles which have the three sides of the one equal to those of the other each to each has been satisfactorily discussed, what further need is there of I. 7, 8? The division of Book I. into distinct sections—the second being on *Parallels*, the third and last on the *Equality of Areas*—is to be commended. In the notes and explanations contained in the second section an approved alternative treatment of parallels is added. Perhaps it may be found that the plan of giving alternatives is, in this particular case, the only compromise likely to gain general acceptance; but the section is incomplete without the proof by superposition—to be found in Mr. Reynolds' work—of I. 27. In the figures of II. 4—8 we are happily emancipated from the diagonals.

On the whole, it may be said, that Mr. Smith has taken a perhaps over-cautious step in the right direction. The impression is gaining ground in some quarters—and the re-action in favour of Euclid on some parts of the Continent tends to confirm it—that our great educational want is a reformed Euclid, as distinguished from a new Geometry. Other Editors have given us good notes in their Appendices, but it is not enough to have improvements in an Appendix; we want them introduced into the text itself, and we are prepared in certain cases to dispense with time-honoured Euclid proofs and propositions altogether. On the other hand, we think that the denunciation of Euclid is sometimes carried to excess. Euclid is often made to appear unnecessarily repulsive, because it is not *taught* and illustrated. Boys are simply set to *learn* it almost word for word, and what text-book can be expected to bear such an ordeal? C. T.

---

* We understand that an additional two months has since been allowed.

BOURDON'S METALLIC BAROMETER.

By *E. Hill, M.A.*, Fellow of St. John's College.

BOURDON'S Metallic Barometer is described in Besant's *Elementary Hydrostatics*, Chap. V. (notes). It consists of a thin elastic metal tube of elliptic section, in shape a portion of a circle, closed at its ends, and exhausted of air. Alterations in the pressure of the atmosphere are indicated by the ends of the tube approaching towards or receding from each other.

I am not aware that any definite explanation of the principle of its action has been offered. A partial explanation current is that it is the principle on which a full sack tends to straighten itself. The meaning of this appears to be, that when the external pressure on the metal decreases, the internal pressure may be considered to increase relatively, and therefore the effect produced will be the same as if air were forced in, and analogous to the result of filling a sack fuller. An increase of external pressure would produce, of course, an opposite effect. This explanation is probably true, but hardly satisfies the mind. One may ask, why should the full sack tend to straighten? and if it be answered that thereby the capacity is increased, still this itself requires proof.

The following discussion may perhaps be found a more complete explanation, and is strictly elementary.

Let $ABA'B'$ (fig. 1) be a section of the tube at right angles to its axis. The curvatures at B, B' are flatter than at A, A', so that an increase of external pressure will squeeze B, B' nearer each other, and force A, A' farther apart.

Let fig. 2 be a section of the tube by a plane parallel the plane of its circle.

Let the radii of the circular arcs CBD, $cB'd$, and the angle COD, be R, r, θ respectively. Let an increase of atmospheric pressure alter them to R', r', θ' respectively. Now we may regard the lengths of CBD, $cB'd$, as unchanged, since steel, while it will bend readily, lengthens only imperceptibly under ordinary forces.

Hence $$R\theta = R'\theta',$$

$$r\theta = r'\theta'.$$

Subtracting $$(R - r)\,\theta = (R' - r')\,\theta'.$$

Now $R-r$, $R'-r'$ are the minor axes of the elliptic section before and after the increase of atmospheric pressure. This increase, squeezing up the ellipse, diminishes the minor axis;

therefore $\qquad R'-r'$ is less than $R-r$;

therefore $\qquad \theta'$ is greater than θ,

that is, θ increases with the pressure. But if θ increases, the ends of the tube, C and D must approach each other. In like manner if the pressure decrease, they will open and recede. And this is what is observed to take place.

It seems to follow from this investigation, that if the major axis of the elliptic section lay in the plane of the circle, an increase of pressure would make the ends of the tube recede instead of approach; and that if the section were circular no effect would be produced.

Let us examine in what form the instrument will be most sensitive.

Calling b, b' the lengths of the semi-minor axis of the elliptic section before and after compression, we have

$$R-r=2b, \quad R'-r'=2b'.$$

Hence from above $\qquad b\theta=b'\theta'.$

Put for b', $b-x$, for θ', $\theta+\alpha$, so that x and α are both small quantities.

Then $\qquad b\theta=(b-x)(\theta+\alpha)$;

therefore $\qquad 0=-x\theta+b\alpha-\alpha x$;

therefore $\qquad x\theta=b\alpha$,

approximately neglecting αx, which, being the product of two small quantities, is very small; therefore

$$\alpha=x\frac{\theta}{b};$$

therefore for a given compression of the ellipse, the sensibility varies as the angle θ. The instrument should therefore be nearly a complete circle. Probably also no serious alteration in the indications would result if the shape were not truly circular. If so, by making it spiral, so that the ends could overlap, it might be lengthened to any extent, and the sensibility greatly increased.

We suppose in this case the change in the angle COD to be used as the indication of the change of pressure. If the change in the arc CD be used, the sensibility will be found

to vary inversely as θ; and if the change in the chord CD, then it will be approximately independent of the angle.

The reader will notice that the elasticity of the steel offers resistance to a change of the circular curvature CBD, as well as of the elliptical, ABA'. If the tube were composed of a number of elliptical steel rings like ABA' joined by oiled silk or some such material that will not stretch and is air tight, into a closed tube of the same shape as this instrument, it would give indications in the same manner; the investigations of this article would all hold, and it would only be necessary to calculate the compression of the ellipse for a given increase of pressure, from which the change in θ follows from the formula above, $\alpha = \frac{\theta}{b} . x$.

So again if the tube were constructed of elastic wire arcs like CBD, parallel to each other, connected by some inelastic but flexible membrane, it would give indications. For a compression of the elliptic section would as above necessitate a bending of the wire arcs, which their elasticity would resist.

The actual case is a combination of both these. The pressure must cause the steel to bend both along the elliptic and circular sections, and the joint resistance to bending must be overcome. The calculation of the actual bending produced by a given increase of pressure appears difficult.

IMPACT OF AN ELASTIC ROD.*

By *J. Hopkinson, D.Sc.*

IN the last Number of the *Messenger*, I offered an explanation of the fact, that the high tones of a rod or a tuning fork die out faster than the lower ones, and that a very short rod will not sound; and a rough explanation of the laws of impact, assuming that the internal friction between two parts of the body impinging is proportional to

* The paper in the last number of the former series of the *Messenger*, "On the Effect of Internal Friction on the Vibrations of a Solid," was published without my knowledge. Shortly after its completion, I found that a considerable portion was not new, I therefore requested that it should not appear; it was however subsequently printed, and the proof sheets were never forwarded to me; the misprints are in consequence so numerous that the paper is somewhat unintelligible, the most serious being that $\frac{d^2\xi}{dx^2}$ s rendered $\frac{d^0\xi}{dx^0}$.

their relative velocity. The same assumption may be better expressed thus: let s be an element of strain, the corresponding element of stress $= As + B\frac{ds}{dt}$ where A and B are constants for the substance, and t is the time. A strain is the change in the relative position of the particles of a body, a stress in the force between the particles called into play thereby.

On this hypothesis let us examine the motion of an elastic rod impinging perpendicularly on a perfectly rigid wall. The time t is measured from the moment of contact, the origin of abscissæ is the point of contact with the wall, x is the distance of a point of the rod from the end when the rod is unstretched, and $x + \xi$ is the abscissa of that point at time t. Let the tension in the rod at this point be $E\left(\frac{d\xi}{dx} + \mu\frac{d^2\xi}{dx\,dt}\right)$; for here the element of strain is $\frac{d\xi}{dx}$. E is Hooke's coefficient of elasticity, and μ depends on the viscosity of the material.

The equation of motion of the rod will be

$$\frac{d^2\xi}{dt^2} = a^2\left(\frac{d^2\xi}{dx^2} + \mu\frac{d^3\xi}{dx^2\,dt}\right) \text{.............} (1),$$

where $$a^2 = \frac{E}{\text{weight of unit of length}},$$

the conditions are, that when $x = 0$, $\xi = 0$, and when $x = l$, $\frac{d\xi}{dx} = 0$, for all values of t till the rod rebounds; and that when $t = 0$, $\xi = 0$, and $\frac{d\xi}{dt} = -V$ for all values of x from 0 to l; l being the length of the rod, and V the velocity of impact.

Before considering equation (1), let us take the case of a perfectly elastic bar where μ vanishes; it might at first sight appear probable that the bar would spring back in a state of vibration, that a portion of the *vis viva* would be converted into vibration of the nature of sound, and that therefore the velocity of recoil would be less than that of impact; in this case it happens not to be so.

When $\mu = 0$, the solution of (1) will be

$$\xi = \Sigma A_n \sin\frac{n\pi}{2}\frac{x}{l}\sin\frac{n\pi}{2}\frac{at}{l} \text{ with } n \text{ odd},$$

and A_n is determined by the equation

$$-V = \Sigma A_n . \frac{n\pi a}{2l} . \sin\frac{n\pi}{2}\frac{x}{l},$$

multiplying by $\sin\frac{n\pi}{2}\frac{x}{l}$, and integrating from $x=0$ to $x=l$, we have

$$-V\frac{2l}{n\pi} = \tfrac{1}{2}A_n . \frac{n\pi a}{2l} . l,$$

or
$$A_n = -\frac{8l}{n^2\pi^2 a} V,$$

our result is then

$$\xi = \frac{-8Vl}{\pi^2 a}\Sigma_1^\infty \frac{1}{n^2}\sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l} \quad\ldots\ldots\ldots\ldots (2),$$

the pressure between the wall and the rod

$$= -E\frac{d\xi}{dx} \text{ when } x=0$$

$$= \frac{4VE}{\pi a}\Sigma \pm \frac{1}{n}\sin\frac{n\pi at}{2l},$$

and this will again vanish when $at = 2l$, throughout the rod at this time $\xi = 0$, and

$$\frac{d\xi}{dt} = V,$$

hence the rod will rebound with the same velocity as that of impact, and not in a state of vibration.

To return to the general equation (1), assume the form of solution

$$\xi = \Sigma A_n E^{-\lambda t} . \sin mx . \sin pt,$$

substitute in equation (1), and equate coefficients of $\sin pt$ and $\cos pt$,

$$(\lambda^2 - p^2) = -m^2a^2 + m^2a^2\mu\lambda,$$

and
$$-2\lambda p = -a^2\mu m^2 p;$$

therefore
$$\lambda = \tfrac{1}{2}a^2m^2\mu,$$

and
$$p^2 = m^2a^2 - \tfrac{1}{4}m^4a^4\mu^2,$$

now m is an odd multiple of $\frac{\pi}{2l}$, and μ is small so that its square may, in this case, be neglected.

We have then

$$\xi = \Sigma A_n \varepsilon^{-\frac{a^2 m^2 \mu t}{2}} . \sin mx . \sin mat \text{ ....} \quad (3),$$

it is easy to see that the values of A_n are the same as before.

The reaction will vanish at the same epoch as before, and at that time $\xi = 0$ throughout the rod, but $\frac{d\xi}{dt}$ is not constant; hence the rod will rebound in a state of vibration, the velocity of each element being given by the equation

$$\frac{d\xi}{dt} = - \Sigma A_n \frac{n\pi a}{2l} \varepsilon^{-\frac{a\pi^2 \mu n^2}{4l}} \sin \frac{n\pi x}{2l},$$

to find the velocity of the rebound or the velocity of the centre of gravity, we must integrate with respect to x and divide by l, we have velocity

$$= - \Sigma A_n \frac{a}{l} \varepsilon^{-\frac{a\pi^2 \mu n^2}{4l}}$$

$$= V \frac{8}{\pi^2} \Sigma \frac{1}{n^2} \varepsilon^{-\frac{a\pi^2 n^2 \mu}{4l}} \text{ ....} \quad (4).$$

This result clearly verifies for the case of $\mu = 0$, for

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

From equation (4) we may conclude, first, that the recoil is proportional to the velocity of impact for the same bar, their ratio being the modulus of elasticity for that bar.

Second, the modulus is greater for a long bar than a short one.

As to whether the latter conclusion is in accordance with experiment, I do not know, but it seems to agree with the fact, that short waves are more quickly destroyed by friction than long ones.

NOTE ON A DYNAMICAL PARADOX.

By *N. M. Ferrers, M.A.*

PROFESSOR STOKES' interesting discussion of his Smith's prize question may perhaps receive further elucidation as follows: In the equation which determines the time of vibration, write x for l', and y for $\frac{g}{n^2}$, or $\frac{gT^2}{\pi^2}$, we then get

$$mlx - (m+m')(l+x)y + (m+m')y^2 = 0;$$

or $$\left(y - \frac{m}{m+m'}l\right)\left(y - x - \frac{m'}{m+m'}l\right) - \frac{mm'}{(m+m')^2}l^2 = 0.$$

Regarding x and y as coordinates, the locus of this equation is an hyperbola, whose asymptotes are represented by the equations

$$y = \frac{m}{m+m'}l,\ y = x + \frac{m'}{m+m'}l,$$

and which passes through the origin. The coordinates of its centre are $\frac{m-m'}{m+m'}l$, $\frac{m}{m+m'}l$.

Since m' is by supposition very small compared with m, the curve is everywhere nearly coincident with its asymptotes.

Its general form will therefore be that represented in fig. 3, in which $OL = l$, and $OM = ON = RL = \frac{m'}{m+m'}l$. An inspection of the figure will shew that for values of n less than $\frac{m-m'}{m+m'}l$, the upper branch of the curve corresponds to values of y nearly $= l$ and the lower to values nearly $= x$. For values of n exceeding $\frac{m-m'}{m+m'}l$ the case is just reversed. This illustrates the fact that where $l' < \frac{m-m'}{m+m'}l$ the greater root of the quadratic approximately corresponds to $T = \pi\left(\frac{l}{g}\right)^{\frac{1}{2}}$ and the less to $T = \pi\left(\frac{l'}{g}\right)^{\frac{1}{2}}$; and *vice versâ* where $l' > \frac{m-m'}{m+m}l$.

NOTE ON LAGRANGE'S DEMONSTRATION OF TAYLOR'S THEOREM.

By Professor *Cayley*.

I TAKE the occasion of the publication of the last edition of Mr. Todhunter's "Treatise on the Differential Calculus" to make some remarks on the demonstration in question. Mr. Todhunter proposes to himself to exhibit a comprehensive view of the Differential Calculus *on the method of Limits;* but he very properly introduces in some cases demonstrations founded upon other views of the subject, pointing out that this is the case, and explaining or indicating his objections. Thus (Chapter VI.) upon Taylor's Theorem, he remarks "Before we offer a strict demonstration of the theorem in question, we shall notice the method which it was usual to adopt in treatises on the Differential Calculus not based on the doctrine of Limits," and then after giving a demonstration depending on the relation $\frac{d}{dx} f(x+h) = \frac{d}{dh} f(x+h)$,* he goes on "There are numerous objections to the method of the preceding articles, and especially the use of an infinite series, without ascertaining that it is convergent, is inadmissible; we proceed then to a rigorous investigation," which investigation (after Mr. Homersham Cox) is a demonstration of the equation

$$f(x+h) = f(x) + hf'(x) \ldots + \frac{h^n}{\underline{|n}} f^n(x) + \frac{h^{n+1}}{\underline{|n+1}} f^{n+1}(x+\theta h).$$

(θ between 0 and 1) whence "if the function $f^{n+1}(x+\theta h)$ is such that by making n sufficiently great the term $\frac{h^{n+1}}{\underline{|n+1}} f^{n+1}(x+\theta h)$ can be made as small as we please, then by carrying on the series

$$f(x) + hf'(x) + \frac{h^2}{\underline{|2}} f''(x) + \frac{h^3}{\underline{|3}} f'''(x) + \ldots$$

to as many terms as we please we obtain a result differing as little as we please from $f(x+h)$. Under these circumstances then we may assert the truth of Taylor's theorem."

* This demonstration is similar in principle to Lagrange's but I think his is preferable: viz. the principle made use of by Lagrange is that the series has the same value whether x is changed into $x + k$, or h into $h + k$.

I share Abel's horror of divergent series,* and I maintain the validity of Lagrange's demonstration. When by an algebraic process we expand a function in a series, for instance the function $\frac{1}{1-x}$, by division

$$\begin{array}{l} 1-x)\ 1 \qquad (1+x+x^2+\&c. \\ \quad\ \ \underline{1-x} \\ \qquad\quad x \\ \qquad\quad \underline{x-x^2} \\ \qquad\qquad x^2\ \&c. \end{array}$$

in the series $1+x+x^2+\&c.$, and write accordingly

$$\frac{1}{1-x}=1+x+x^2+\&c.$$

all that is (or ought to be) meant is that the algebraical operations continued as far as we please will give the series of terms $1, x, x^2, \ldots$ or say the series of coefficients $1, 1, 1, \ldots$ And of course with this meaning of the equation, the objection "*non constat* that the series is convergent" would be wholly irrelevant, we do not say that it is, we do not care whether it is so or not. In further illustration, remark that we frequently use such an equation merely as the means of expressing the law of a series of numbers $a_0, a_1, a_2 \ldots$ say $a_n = \text{coeff. } x^n$ in $f(x)$, where the function is assumed to be by a definite process expansible in the form $a_0 + a_1x + a_2x^2 + \&c.$ in question. Any objection that the series is not convergent would be simply irrelevant. Now any rational or irrational algebraic function $f(x+h)$ can by ordinary algebraical processes be expanded in the form $f(x)+$ terms in h, h^2 &c... And if in regard to a function $f(x)$ we make the *single assumption* that $f(x+h)$ is *expansible in a form containing powers of h, and which reduces itself to $f(x)$ when h is put* $=0$, then Lagrange's demonstration shows that the powers of h are h, h^2, h^3 &c... and that the expansion in fact is

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{1.2}f''(x)+\&c.$$

viz. $f(x+h)$ acquires the same value $f(x+h+k)$ whether

---

* Peut-on imaginer rien de plus horrible que de débiter

$$0=1^n-2^n+3^n-4^n+\ldots$$

où n est un nombre entier positif?—*Œuvres*, t. II., p. 266.

we change therein x into $x+k$ or h into $h+k$; and the expression on the right-hand side is the only series in h possessed of the same property. It is to be remarked that the equation contains in itself the definition of the operation of derivation, viz., the equation being true, $f'(x)$ can only denote the coefficient of h in the expansion of $f(x+h)$; and what really is shown is that admitting that such an operation is possible in regard not only to $f(x)$, but to $f'(x)$ &c., then that the coefficients $f'(x), \frac{f''(x)}{1.2}$ &c., are obtained from $f(x)$ by the successive repetitions of this operation and by dividing by the proper numerical denominator.

By what precedes, any objection in regard to convergency, I regard as irrelevant; and if it is said that the above-mentioned single assumption is not granted, I would either ask "What is a function"—or I would content myself with the hypothetical statement—*if* $f(x)$ be such that $f(x+h)$ is expansible *ut suprà*, then Taylor's theorem.

In regard to the demonstration given by Mr. Todhunter it implicitly assumes that x and h are both real, and (although doubtless possible) it would be considerably more difficult to find an analogous demonstration of the formula involving $f^{n+1}(x+\theta h)$ in the case of x and h imaginary. But the formula *with the term in question* is not (nor does Mr. Todhunter consider as being) Taylor's theorem; to obtain from it Taylor's theorem, we require (in the foregoing point of view) the property that $h^{n+1} f(x+\theta h)$ is expansible in a series involving h^{n+1} and the higher powers of h, that is the very property that $f(x+h)$ is expansible in positive powers of h.

Moreover admitting that the formula with the term $f^{n+1}(x+\theta h)$ is demonstrable for imaginary values of x, h, the formula is *meaningless* in the case where x, h are one or both a symbol or symbols of operation : θ would certainly have no definable numerical magnitude, and if it is considered as meaning anything, then the equation in question is a mere definition of what it does mean, and ceases to be a theorem in regard to $f(x+h)$. It is impossible, in a quantitative algebra such as is presupposed in the method of limits, to put any meaning on the equation

$$f\left(\frac{d}{dx}+h\right)=f\left(\frac{d}{dx}\right)+hf'\left(\frac{d}{dx}\right)+\text{\&c.},$$

which however I regard as a legitimate particular form of Taylor's theorem.

ON THE HISTORY OF EULER'S CONSTANT.

By *J. W. L. Glaisher, B.A.*

Of all the mathematical constants π, the ratio of the circumference of a circle to its diameter and e the base of the Napierian logarithms are unquestionably the most remarkable, not only on account of their frequent use, but also from their connection with mathematical history: next, however, both in interest and importance must be placed Euler's constant ·57721566..., usually defined as the limit of $1+\frac{1}{2}...+\frac{1}{x}-\log x$, when x is infinite.

The value of π has engaged the attention of many mathematicians and calculators from the time of Archimedes to the present day, and has been computed from so many different formulæ, that a complete account of its calculation would almost amount to a history of mathematics. For every practical purpose π to twenty decimal places is as useful as π to five hundred, but nevertheless, every increase in the number of places has been attended with great interest, not so much because the additional figures were of importance, as because each extension has been in general due to some improvement in the method or formula of which the value was obtained; and thus by examining the different determinations we can trace the transition from the geometrical method of Archimedes to the infinite series of recent sines, and it by no means necessarily follows that the 530 places of Shanks and Rutherford in 1853 represent more labour than did the 32 places of Van Ceulen in 1619.* Of scarcely inferior interest is the calculation of e, connected as it is with the whole history of logarithms. Euler's constant (which throughout this note will be called γ after Mascheroni, De Morgan, &c.), though of far less celebrity than π or e, has still strong claims to notice; it was introduced at one of the most remarkable periods in mathematical history, and although originally merely connected with the harmonic series, it has since acquired importance, both from its connection with the Gamma Function, and its occurrence in the expansions of the cosine-integral, exponential-integral and logarithm-integral. Additional interest has also been conferred on the history of the

---

* Prof. de Haan expressly states, that the earlier calculations were quite as laborious as those of recent times. *Amsterdam Transactions*, Vol. IV.

constant, by the fact that, owing to a miscalculation of Mascheroni, two values of it have been in existence, and this has occasioned much uncertainty as to its true value, which has only been removed within the last fourteen years by the independent calculations of Lindman and Oettinger.

The constant is now best known to mathematicians from its occurrence in the formula

$$1+\frac{1}{2}\ldots+\frac{1}{x}=\gamma+\log x+\frac{1}{2x}-\frac{B_1}{2x^2}+\frac{B_2}{4x^4}-\frac{B_3}{6x^6}+\ldots\ldots\ (1),$$

B_1, B_2... being Bernoulli's numbers, but although its introduction by Euler was subsequent to Bernoulli's discussion of the numbers that bear his name, it was not until after the lapse of many years that the latter became sufficiently well known to lead to the summation of the harmonic series in the form (1).

During the first half of the last century, the theory of infinite series began to excite great attention, and numerous memoirs of Euler and the Bernoullis were written on the subject. The series $1+\frac{1}{2^2}+\frac{1}{3^2}\ldots$ ('the despair of analysts' as Montucla calls it) for many years exercised in vain the powers of mathematicians, until it was at last successfully summed by Euler in 1748. The harmonic series $1+\frac{1}{2}+\frac{1}{3}\ldots$ also, remarkable as being the only divergent series of the group $1+\frac{1}{2^n}+\frac{1}{3^n}\ldots$, received no small share of attention about this time.

The discovery that the series $1+\frac{1}{2}+\frac{1}{3}\ldots$ was divergent, is attributed by James Bernoulli to his brother (*Ars Conjectandi*, p. 250), but the connection between $1+\frac{1}{2}\ldots+\frac{1}{x}$ and $\log x$ was first established by Euler ("De Progressionibus harmonicis observationes," *Comm. Acad. Petropol.* t. VII. for 1734 and 1735, p. 156) as follows:

$$1=\log 2 \quad +\frac{1}{2} \quad -\frac{1}{3} \quad +\frac{1}{4} \quad -\frac{1}{5} \quad +\ldots$$

$$\frac{1}{2}=\log\frac{3}{2} \quad +\frac{1}{2}\cdot\frac{1}{2^2}-\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{4}\cdot\frac{1}{2^4}-\frac{1}{5}\cdot\frac{1}{2^5}+\ldots$$

$$\vdots$$

$$\frac{1}{i}=\log\frac{i+1}{i}+\frac{1}{2}\cdot\frac{1}{i^2}-\frac{1}{3}\cdot\frac{1}{i^3}+\frac{1}{4}\cdot\frac{1}{i^4}-\frac{1}{5}\cdot\frac{1}{i^5}+\ldots,$$

and thence by addition

$$1+\frac{1}{2}\ldots+\frac{1}{i}=\log(i+1)+\frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}\ldots\right)$$
$$-\frac{1}{3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}\ldots\right)+\frac{1}{4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}\ldots\right)\ldots\ldots(2).$$

"Quæ series," Euler proceeds, "cum sint convergentes, si proxime summentur prodibit

$$1+\frac{1}{2}\ldots+\frac{1}{i}=\log(i+1)+0{\cdot}577218.$$

Si summa dicatur s, foret, ut supra fecimus, $ds=\frac{di}{i+1}$, ideoque $s=\log(i+1)+C$. Hujus igitur quantitatis constantis C valorem deteximus, quippe est $C=0{\cdot}577218$." This is, I believe, the first mention of the constant in mathematics.

In the same transactions, for 1769 (t. XIV., Part I., p. 153) in a paper "De summis serierum numeros Bernoullianos involventium," (which contains Euler's well-known values of the first seventeen Bernoulli's numbers) Euler returns to the harmonic series and gives the formula marked (1) in this note, from which by putting $x=10$ he calculates $\gamma={\cdot}5772156649015325\ldots$. He also gives another formula very similar to (2) viz.

$$\gamma=\frac{1}{2}\left(\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)+\frac{2}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}+\ldots\right)+\frac{3}{4}\left(\frac{1}{2^4}+\frac{1}{3^4}+\ldots\right)+\ldots \ldots\ldots\ldots(3),$$

and concludes this portion of his memoir by the remark "Manet ergo quæstio magni momenti, cujusdam indolis sit numerus iste γ, et ad quodnam genus quantitatum sit referendus."

How strongly Euler felt this desire to connect γ with some other known constant, or to discover some property attaching to it, is evident from a memoir, having the discussion of this constant for its sole object, which he communicated to the Petersburgh Academy in 1781.* Near the commencement of this paper Euler speaks of γ as "magis notatu dignus, quod eum nullo adhuc modo ad quampiam mensuram cognitam revocare mihi quidem licuit." As the formula (1) however when x is very great becomes $1+\frac{1}{2}\ldots+\frac{1}{x}=\gamma+\log x$,

* "De numero memorabili in summatione progressionis harmonicæ naturalis occurrente," *Acta Petrop*, t. v., Part II., p. 45.

he thinks it probable that γ may be the logarithm of some remarkable number, so that if $\gamma = \log N$ we should have $\gamma + \log x = \log (Nx)$. Having met with no success in endeavouring to connect N with any other constant, Euler proceeds to investigate some new formulæ for γ, giving as his reasons the complicated law which Bernoulli's numbers follow and the uncertainty which might attach to the calculation of γ from (1), since that series is ultimately divergent.

The new formulæ obtained in this memoir are

$$1-\gamma = \frac{1}{2}(s_2-1)+\frac{1}{3}(s_3-1)+\frac{1}{4}(s_4-1)+\dots \quad \dots(4),$$

$$2\gamma - 1 = \left(\frac{1}{2}+\frac{1}{3}-\frac{2}{3}s_2\right)+\left(\frac{1}{4}+\frac{1}{5}-\frac{2}{5}s_4\right) + \left(\frac{1}{6}+\frac{1}{7}-\frac{2}{7}s_6\right)+\dots(5),$$

$$2-2\log 2-\gamma = \left(\frac{2}{3}\cdot\frac{7}{8}s_2-\frac{2}{3}\right)+\left(\frac{2}{5}\cdot\frac{31}{32}s_4-\frac{2}{5}\right) + \left(\frac{2}{7}\cdot\frac{127}{128}s_6-\frac{2}{7}\right)+\dots(6),$$

$$1-2\log 2+\gamma = \left(\frac{1}{2}-\frac{1}{3}-\frac{2}{3.2^3}s_2\right)+\left(\frac{1}{4}-\frac{1}{5}-\frac{2}{5.2^5}s_4\right)+\dots(7),$$

$$\log 2-\gamma = \frac{1}{3.2^2}s_2+\frac{1}{5.2^4}s_4+\frac{1}{7.2^6}s_6+\dots \quad \dots\dots\dots\dots(8),$$

$$1-\log\tfrac{3}{2}-\gamma = \frac{1}{3.2^2}(s_2-1)+\frac{1}{5.2^4}(s_4-1)+\frac{1}{7.2^6}(s_6-1)\dots(9),$$

where $$s_n = 1+\frac{1}{2^n}+\frac{1}{3^n}+\dots$$

From (4) Euler calculates γ to five places correctly and from (9) to twelve places.

Mascheroni, in his *Adnotationes ad Euleri Calculum Integralem*,* subsequently extended the calculation to 32 figures as follows:

$$\gamma = {\cdot}57721\ 56649\ 01532\ 86061\ 81120\ 90082\ 39\dots$$

---

* This work I have not seen, it is referred to by Lacroix (*Calc. Diff. et Int.* t. III., p. 521).

In 1809, Soldner published at Munich his "Théorie d'une nouvelle fonction transcendante," the new transcendent being the logarithm-integral

$$li\,x = \int_0 \frac{dx}{\log x} = \gamma + \log\log x + \log x + \frac{(\log x)^2}{2^2} + \ldots;$$

the value of γ Soldner gives on p. 13 as

$$\gamma = 57721\ 56649\ 01532\ 86060\ 6065\ldots$$

which differs from Mascheroni's value in the twentieth place.

This want of agreement led Gauss,* in 1812, to urge F. G. B. Nicolai, "juvenem in calculo indefessum," to undertake the calculation and decide which was the true value. Nicolai used formula (1) and obtained γ correctly to forty places, both from $n = 50$ and $x = 100$. This double calculation, which confirmed Soldner's value, set the matter completely at rest; but, unfortunately, the note in Gauss's memoir, which contains the result of Nicolai's calculation, seems to have attracted little notice, and Mascheroni's value has been repeatedly quoted since; thus it is given by Lacroix (*Calc. Diff. et Int.*, t. III., p. 521); by Bretschneider (*Crelle*, t. 17, p. 260); by Bessel (*Königsberger Archiv*, p. 4); and in Grunert's supplement to Klügel's *Wörterbuch* both values are given, the author having clearly not been aware of Nicholai's calculations.

The occurrence of the two values in Klügel induced Lindman to recalculate the constant, and his values are given in Grunert's *Archiv*† for 1857. He used formula (1) and obtained γ correctly to 34 places from $x = 100$ and to 24 places from $x = 20$. Four years later, in ignorance of what Lindman had done, Oettinger‡ applied himself to the investigation of the true value, which he verified by four different calculations; using formula (1) he obtained γ correctly to 18, 25, 34, and 41 places, by giving x successively the values 10, 20, 50, and 100.

Thus it will be seen that Mascheroni's error has led to eight additional calculations of the constant.

It should be mentioned that Bretschneider in giving

* Disquisitiones generales circa seriem infinitam $1 + \frac{\alpha\beta}{\gamma}x + \ldots$ *Comment. Soc. Reg. Gotting*, t. II., see the note p. 36 of the memoir.

† "De vero valore constantis quæ in logarithmo integrali occurrit," t. 29, p. 238.

‡ "Ueber die richtige Werthbestimmung der Constante des Integrallogarithmus," *Crelle*, t. 70, p. 375.

Mascheroni's value (*Crelle*, t. 19, p. 260) attributes it to Kramp. This is most probably a mistake, as both Lacroix and Gauss refer it to Mascheroni; and Bretschneider himself, in alluding to the error (*Zeitschrift für Mathematik und Physik*, t. 6, p. 131) speaks of it as Mascheroni's calculation; it would, however, have been more satisfactory if he had either corrected or confirmed definitely his assertion about Kramp. I may mention that I have made a cursory examination of all Kramp's papers given in the Royal Society's Catalogue which seemed to have any chance of containing it, as well as his treatise on Refraction, without success.

On account of its occurrence in the formula

$$\log\Gamma(1+x) = -\gamma x + \tfrac{1}{2}s_2x^2 - \tfrac{1}{3}s_3x^3 + \tfrac{1}{4}s_4x^4 - \ldots$$

$$\log\Gamma(1+x) = \tfrac{1}{2}\log\left(\frac{\pi x}{\sin\pi x}\right) - \gamma x - \tfrac{1}{3}s_3x^3 - \tfrac{1}{5}x^5 - \ldots$$

Legendre* was naturally concerned with the constant and calculated it to 19 places by (1) from $x=10$; he also made two other calculations to 15 places, the one from the formula

$$1-\gamma-\tfrac{1}{2}\log 2 = \tfrac{1}{3}(s_3-1) + \tfrac{1}{5}(s_5-1) + \ldots\ldots(10),$$

and the other from that marked (9) in this note, chiefly with the view of verifying the values of $s_1, s_2 \ldots$.

The most recent determinations of the constant are those of Mr. Shanks (*Proc. Roy. Soc.*, t. 15, p. 429), who has calculated it from (1) to 21, 28, 39, 46, 54, 59, and 59 places from $x = 10$, 20, 50, 100, 200, 500, and 1000 respectively. The value (after correction of an error in the fiftieth place) to 59 places is

$$\gamma = \cdot 57721\ 56649\ 01532\ 86060\ 65120\ 90082\ 40243\ 10421$$
$$59335\ 93992\ 35988\ 0577\ldots.$$

It is clearly convenient that the constant should generally be denoted by the same letter. Euler used C and O for it; Legendre, Lindman, &c., C; De Haan A; and Mascheroni, De Morgan, Boole, &c., have written it γ, which is clearly the most suitable, if it is to have a distinctive letter assigned to it. It has sometimes (as in *Crelle*, t. 57, p. 128) been quoted as Mascheroni's constant, but it is evident that Euler's labours have abundantly justified his claim to its being named after him.

---

* *Traité des Fonctions Elliptiques*, t. II., p. 434.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

At a Meeting of this Society, held Thursday, May 11th, at 8 P.M., W. Spottiswoode, Esq., F.R.S., President, in the chair, Mr. C. J. Monro, B.A., late Fellow of Trinity College, Cambridge, was elected a Member. Mr. J. Griffiths, M.A., Fellow of Jesus College, Oxford, was proposed for election, and the Hon. J. W. Strutt, Fellow of Trinity College, Cambridge, was admitted into the Society.

Professor Henrici, V.P., indicated the method of treatment he had employed in his paper "On the Singularities of the Envelopes of a Non-unicursal Series of Curves." Prof. Cayley made a few remarks upon the paper. The Hon. J. W. Strutt then read his paper "On the Resultant of a large number of Vibrations of Irregular Phase, as applied to the explanation of Coronas." This note consisted of an explanation of coronas as given in Verdet's Leçons d'Optique Physique and other works; and the author pointed out what he considered to be a fallacy in the proof as given in Billet's Traité d'Optique Physique, tom. i. p. 235. Sir W. Thomson, Prof. Maxwell, and Mr. Strutt made some further remarks on the subject of the paper. Mr. Maxwell then gave a description of two singular solar halos which he had recently seen; and Prof. Adams, of King's College, gave some additional particulars in the case of one of the phenomena which had also been noticed by himself. Prof. Cayley V.P., communicated an account of a paper by Mr. Griffiths, of Jesus College, Oxford, "On the problem of finding the circle which cuts three given circles at given angles." The President next requested assistance in the solution of a "Question in the Mathematical Theory of Vibrating Strings," which he had been unable to solve. A string is said to execute a forced vibration when it is compelled to perform vibrations synchronous with those of a vibrating body to which one end is attached. The amplitude of the forced vibrations is greatest when the length or tension of the string is so adjusted that its natural period of vibration, for its fundamental note or one of its harmonics, is the same as that of the forcing body, and, within limits, the amplitude diminishes as the period of the string diverges from that of the body. The theory of forced vibrations when the motion of the forcing body is transverse to the length of the string has been fully discussed by various writers, and especially by Helmholtz and Dordini. The latter of these has shewn how it is that the amplitude of vibration of the string may vary while that of the body remains constant; and further, he has given expressions which do not lead to the absurd conclusion (resulting from the ordinary formulæ) of an infinite amplitude when the periods of string and body are absolutely synchronous. The expressions in question, or others easily deducible from them, when applied to the discussion of the nodes and length of vibrating segments, explain the phenomena which may be observed in experiment, of nodes of least motion rather than of perfect rest, and of segments of varying length.

But of the forced vibrations, when the forcing body moves in the direction of the string's length, there is, so far as the speaker was aware, as yet no mathematical theory. The principal phenomenon is well known, viz., that when the length and tension of the string are such that it gives out the same note as the forcing body if the forcing motion is transverse it will give out the octave below when the forcing motion is longitudinal.

In this problem there arise two *primâ facie* difficulties—first, that a forcing motion may be conceived, and indeed may be experimentally set up, without producing a vibration in the string; and secondly, that a mechanical vibration of a given period can give rise to a vibration of a period double of the former. The key to the explanation of these difficulties is probably to be found in the consideration, corroborated by experience, that the motion of the string depends upon small quantities of the second order. A mathematical solution of the question is the desideratum suggested.

Mr. Strutt made a few remarks on the subject, and mentioned some results he had arrived at. A communication from Prof. Cayley respecting the extension of the Society's sphere of action was laid before the Meeting by the President;

it was determined that the matter should be discussed at the next meeting of the Society. Prof. Clerk Maxwell asked for information from the Members as to the convention established among mathematicians with respect to the relation between the positive direction of motion along any axis and the positive direction of rotation round it. In Sir W. R. Hamilton's Lectures on Quaternions the coordinate axes are drawn x to South, y to West, and z upwards. The same system is adopted in Prof. Tait's Quaternions and in Listing's Vorstudien Zür Topologie. The positive directions of translation and of rotation are thus connected in a left-handed screw on the tendril of the hop. On the other hand in Thomson's and Tait's Natural Philosophy, § 234, the relations are defined with reference to a watch, and lead to the opposite system, symbolized by an ordinary or right-handed screw, or the tendril of the vine. If the actual rotation of the Earth from West to East be taken positive the direction of the Earth's axis from South to North is positive in this system. In pure mathematics little inconvenience is felt from this want of uniformity, but in astronomy, electro-magnetics, and all physical sciences it is of the greatest importance that one or other system should be specified and persevered in. The relation between the one system and the other is the same as that between an object and its reflected image, and the operation of passing from the one to the other has been called by Listing *Perversion*.

Sir W. Thomson and Dr. Hirst stated the arguments in favour of the right-handed system, derived from the motion of the earth and planets, and the convention that North is to be reckoned positive, and also from the practice of mathematicians in drawing x to the right hand and y upwards on the plane of the black board, and z towards the spectator. No arguments in favour of the opposite system being given, the right-handed system symbolized by a corkscrew or the tendril of the vine was adopted by the Society.

Several presents were made to the Society's Library.

R. TUCKER, M.A., *Hon. Sec.*

Cambridge Philosophical Society.

March 13.—Prof. Adams made a communication to the Society "On the attraction of an infinitely thin shell bounded by two similar and similarly situated concentric ellipsoids on an external point." (Of this paper we hope to give some account in the next number.) Papers were also read by Mr. W. P. Hiern "On a theory of the forms of floating leaves in certain plants," in which he showed that the form of the portion of the margin at any time has either

$\tan\left(\frac{s}{l}\cos\alpha\right) = \cos\alpha\tan\phi$ or $e^{\frac{2s}{l}\cot\beta} = \frac{\sin(\beta+\phi)}{\sin(\beta\sim\phi)}$ for its intrinsic equation; and by Mr. Moon "On the effect of exhaustion and inflation of the tympanum in deadening sounds, and on the test of loudness."

May 1.—Mr. Todhunter, F.R.S., made a communication to the Society "On the measurement of an arc of the meridian in Lapland," in which the author drew attention to the numerous errors which had been made, even by distinguished astronomers, in their accounts of the two measurements of an arc of the meridian in Lapland. A comparison of the original authorities on the subject at once detects these errors and supplies the necessary corrections.

May 14.—The following communications were made to the Society: "On Dr. Wiener's model of a cubic surface with 27 real lines" and "On the construction of a double sixer," by Prof. Cayley. "On the tides in a rotating globe covered by a sea of constant depth at all points in the same latitude, and attracted by a moon always in the plane of the equator, supposed either fixed or moving with uniform angular velocity; considered with reference to the tides as they are in nature and the retardation of the earth's angular motion," and "On the motion of an imperfect fluid in a hollow sphere rotating about its centre under the action of impressed external periodic forces, considered with reference to the phenomena of Precession and Nutation," by Mr. Röhrs. In the former paper the author obtained as his result that the friction of the tides could not increase the length of the day by more than one second in a hundred million years. And the results of the second paper pointed to a condition of the interior of the globe, which was inconsistent with all notions of fluidity.

QUERY. Mr. Tucker sends the following Query:—Are there any fairly simple geometrical constructions for drawing common tangents to a pair of conics?

ON MAXIMA AND MINIMA.

By *R. W. Genese, B.A.*, Scholar of S. John's College, Cambridge.

THE object of this paper is to show how the Maxima or Minima of algebraical expressions of any degree may be determined without the use of the Calculus. I was led to the investigation in the following way: It was proposed to me to find the maxima or minima of $x^3 - 3ax^2$ by an elementary method. It was obvious that (by the geometrical method referred to by me in a former paper) I had only to determine for what values of x the tangent to the curve

$$y = x^3 - 3ax^2$$

was parallel to the axis of x.

Or, again, for what value of y there would be two coincident values of x.

Now

$$x^3 - 3ax^2 - y \div (x-\alpha)^2 = x + (2\alpha - 3a) + \frac{(3\alpha^2 - 6a\alpha)x - 2\alpha^3 + 3a\alpha^2 - y}{(x-\alpha)^2}.$$

Thus if $\quad 3\alpha^2 - 6a\alpha = 0$ (i.e. $\alpha = 0$ or $2a$),

and $\quad y = 3a\alpha^2 - 2\alpha^3$;

then will $\quad x^3 - 3ax^2 - y = (x-\alpha)^2 \{x + (2\alpha - 3a)\}$.

Hence $x = 0$ or $2a$ gives $y = 0$ or $-4a^3$ as maximum or minimum values of $x^3 - 3ax^2$.

I shall not stop over the discrimination, which is easy.

The above method was clearly general, thus, in order to determine the maxima or minima values of

$$y \equiv x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$$

I had only to determine y so that

$$U \equiv x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n - y$$

should have a factor $(x-\alpha)^2$.

Now $\dfrac{U}{x-\alpha} = x^{n-1} + (\alpha + a_1) x^{n-2} + (\alpha^2 + a_1\alpha + a_2) x^{n-3} + \ldots$

and there is no remainder if $y = \alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + \ldots + a_n$.

If $\dfrac{U}{x-\alpha}$ be again divisible by $x - \alpha$ it is clear that on putting $x = \alpha$ we should get zero, thus,

$$0 = n\alpha^{n-1} + (n-1)\, a_1\alpha^{n-2} + \ldots + a_{n-1}.$$

This equation (the "first derived" of $y=0$) will clearly give all the values of α for which $U \equiv (x-\alpha)^2 f(x)$: and therefore, &c.

It would be tedious to explain geometrically the difficulty which arises when the value of y gives three or more equal values of x. I shall at once proceed to the algebraical method suggested by the above.

If y be a maximum or minimum y − constant will be also; if this constant $= \alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + \ldots + a_n$,

then $y - C = x^n - \alpha^n + a_1(x^{n-1} - \alpha^{n-1}) + a_2(x^{n-2} - \alpha^{n-2}) + \ldots$,

therefore
$$\frac{y-C}{x-\alpha} = x^{n-1} + \alpha x^{n-2} + \alpha^2 x^{n-3} + \ldots;$$
$$+ a_1\{x^{n-2} + \alpha x^{n-3} + \ldots\}$$
$$+ a_2\{x^{n-3} + \alpha x^{n-4} + \ldots\}$$
$$+ \&c.$$

$\frac{y-C}{x-\alpha}$ will again be divisible by $x-\alpha$ if
$$0 = n\alpha^{n-1} + (n-1)\,a_1\alpha^{n-2} + \ldots.$$

Let us call this "the auxiliary equation," and let its roots be $\alpha_1\ \alpha_2 \ldots \alpha_{n-1}$, thus
$$y - C_1 = (x-\alpha_1)^2 f_1(x),$$
$$y - C_2 = (x-\alpha_2)^2 f_2(x),$$
$$\&c.$$

where $C_1\ C_2$ are the corresponding values of C, &c.

Let us take as the *type* of these equations
$$y - C = (x-\alpha)^2 f(x)$$

and let the highest power of $(x-\alpha)$ *contained in* $f(x)$ *be* $(x-\alpha)^p$ (p may have any value from 0 to $n-2$).

Thus
$$y = C + (x-\alpha)^{p+2}\,\psi(x)$$

when $\psi(x)$ is not divisible by $x-\alpha$, and therefore does not vanish when $x=\alpha$.

Then when $x=\alpha$, $y=C$, and

1° If p be even, as x passes through the value α, y is always $\left.\begin{matrix}\text{greater}\\ \text{less}\end{matrix}\right\}$ than C according as $\psi(\alpha)$ is $\left.\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ thus there is here a minimum or maximum value of y.

2° If p be odd, as x passes through the value α, y is first greater and next less than C, or *vice versâ*, and there is no maximum or minimum.

Manchester Grammar School,
April 8, 1871.

NOTE ON THE INTEGRAL $\int_0^\infty \frac{\cos bx}{a^2+x^2}\,dx$.

By *J. W. L. Glaisher*, B.A.

On p. 242, vol. v. of the former series of the *Messenger of Mathematics* it is pointed out that the ordinary evaluation of this integral by a double differentiation with regard to b is unsound; and although the manner in which it may be made rigorous is there indicated, still, when so corrected it loses its chief claim to notice, viz. its brevity. The following investigation depending on differentiation with regard to a is nearly as short as the other, and, I believe, free from objection.

Let $u = \int_0^\infty \frac{a\cos bx}{a^2+x^2}\,dx$

$$= \left[\frac{a}{b}\cdot\frac{\sin bx}{a^2+x^2} - \frac{a}{b^2}\cdot\frac{2x\cos bx}{(a^2+x^2)^2}\right]_0^\infty - \frac{1}{b^2}\cdot\int_0^\infty \cos bx\,\frac{d^2}{dx^2}\left(\frac{a}{a^2+x^2}\right)dx$$

by a double integration by parts,

$$= -\frac{1}{b^2}\int_0^\infty \cos bx\,\frac{d^2}{dx^2}\left(\frac{a}{a^2+x^2}\right)dx = \frac{1}{b^2}\int_0^\infty \cos bx\,\frac{d^2}{da^2}\left(\frac{a}{a^2+x^2}\right)dx,$$

whence $\frac{d^2u}{da^2} = b^2u$ and therefore $u = Ae^{ab} + Be^{-ab}$; from this by putting successively $b=\infty$ and $b=0$, we deduce as usual $A=0$, $B=\frac{\pi}{2}$, (b positive) so that $\int_0^\infty \frac{\cos bx}{a^2+x^2}\,dx = \frac{\pi}{2a}e^{-ab}$, or $\frac{\pi}{2a}e^{ab}$ according as b is positive or negative.

That $\frac{d^2}{dx^2}\left(\frac{a}{a^2+x^2}\right) = -\frac{d^2}{da^2}\left(\frac{a}{a^2+x^2}\right)$ is easily proved by performing the differentiations, but it is seen at a glance by writing $\frac{a}{a^2+x^2}$ in the form

$$\frac{1}{2}\left(\frac{1}{a+xi} + \frac{1}{a-xi}\right),$$

where $i = \sqrt{(-1)}$.

Both the differentiations with regard to a are clearly unobjectionable, as neither of the expressions so obtained under the integral sign can become infinite or indeterminate for any value of x.

NOTE ON TAYLOR'S THEOREM.

By *M. M. U. Wilkinson, M.A.*

THE following remarks are suggested by Prof. Cayley's interesting note on page 22.

An equation either asserts the equality of two ratios, or that two distinct operations give the same result; and any use of the expression "=" is faulty which contradicts the axiom, "things which are = the same thing are = one another." So the equation,

$$\frac{1-x^r}{1-x}=1+x+x^2+\ldots+x^{r-1}$$

is true, whether x be a ratio or an operation, for all positive integral values of r. But the equation,

$$\frac{1}{1-x}=1+x+x^2+\ldots+x^{r-1},$$

is only true provided $\frac{x^r}{1-x}=0$. Consequently the equation,

$$\frac{1}{1-x}=1+x+x^2+\ldots\ \infty \text{ terms},$$

is only true provided $\frac{x^\infty}{1-x}=0$, and can only be asserted of those ratios which are intermediate to $\left(-1+\frac{c}{\infty}\right)$ and $\left(1-\frac{c}{\infty}\right)$, for all finite values of c, and of those operations which, when performed an infinite number of times, give zero for the result.

No proposition can ever be proved concerning imaginary quantities; attempted proofs generally assuming that every rational integral algebraical equation has a root, a proposition which cannot be proved of the equation, $x^2+1=0$. But we can show that

$$f\{x+h+t(x'+h')\}+f\{x+h-t(x'+h')\}$$
$$=f(x+tx')+f(x-tx')+(h+th')f'(x+tx')+(h-th')f'(x-tx')+\ldots$$
$$+\frac{(h+th')^{n+1}}{\lfloor n+1}f^{n+1}(x+tx'+\theta)+\frac{(h-th')^{n+1}}{\lfloor n+1}f^{n+1}(x-tx'+\theta'),$$

and similarly,

$$\frac{f\{x+h+t(x'+h')\}-f\{x+h-t(x'+h')\}}{t}=\ldots$$

and so the sum of a series, every term of which is real, can often be conveniently expressed in an imaginary form.

Objections to non-convergent series of ratios seem to me to apply equally to non-convergent series of operations. All series of ratios are non-convergent in which any ∞ th term is not evanescent; and so all series of operations are non-convergent in which the result of the ∞ th term in the series of operations is not zero.

SOLUTIONS OF A SMITH'S PRIZE PAPER FOR 1871.

By Professor *Cayley*.

1. *A point moves in a plane with a given velocity, and also with a given velocity about a fixed point in the plane: shew that the locus is either a circle passing through the fixed point, or else a circle having the fixed point for its centre; and explain the relation between the two solutions.*

We have in general

$$v^2=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2,$$

and in the present question, taking the fixed point as the origin, and measuring θ from any fixed line through this point

$$\frac{d\theta}{dt}=\omega,\quad V^2=\left(\frac{dr}{dt}\right)^2+r^2\omega^2,$$

where V, ω are given constants. Hence

$$\left(\frac{dr}{d\theta}\right)^2=\left(\frac{dr}{dt}\right)^2\div\left(\frac{d\theta}{dt}\right)^2=\frac{V^2}{\omega^2}-r^2,$$

or, writing $V = a\omega$,

$$\left(\frac{dr}{d\theta}\right)^2 = a^2 - r^2,$$

therefore
$$d\theta = \frac{dr}{\sqrt{(a^2-r^2)}},$$

or $$\theta + \beta = \sin^{-1}\frac{r}{a} \quad (\beta \text{ the constant of integration}),$$

that is
$$r = a \sin(\theta + \beta),$$

which is the equation of a circle (radius $= \frac{1}{2}a$) passing through the fixed point. In fact, the point moving in such a circle with a constant velocity, moves about the centre with a constant angular velocity, and about any fixed point in the circumference with an angular velocity which is one-half of that about the centre, and is therefore also constant.

Treating β as a variable parameter, to obtain the envelope we have

$$0 = a \cos(\theta + \beta),$$

that is $\theta + \beta = \frac{\pi}{2}$ and therefore $r = a$, which is the equation of a circle (radius $= a$) having the fixed point for its centre. This is consequently the singular solution.

2. *Determine the system of curves which satisfy the differential equation*

$$dx\,\{\sqrt{(1+x^2)} + ny\} + dy\,\{\sqrt{(1+y^2)} + nx\} = 0\,;$$

and shew that the curve which passes through the point $x = 0$, $y = n$ *contains as part of itself the conic*

$$x^2 + y^2 + 2xy\sqrt{(1+n^2)} - n^2 = 0.$$

The equation is integrable *per se*, viz. we have

$$x\sqrt{(1+x^2)} + \log\{x + \sqrt{(1+x^2)}\}$$
$$+ y\sqrt{(1+y^2)} + \log\{y + \sqrt{(1+y^2)}\} + 2nxy = C,$$

or determining the constant so that for $x = 0$ y may be $= n$,

$$C = n\sqrt{(1+n^2)} + \log n + \sqrt{(1+n^2)},$$

and the equation may be written

$$x\sqrt{(1+x^2)} + y\sqrt{(1+y^2)} + 2nxy - n\sqrt{(1+n^2)}$$
$$+ \log\frac{\{x+\sqrt{(1+x^2)}\}\,\{y+\sqrt{(1+y^2)}\}}{n+\sqrt{(1+n^2)}} = 0,$$

which is evidently a transcendental curve; it may however be shown that if $x^2+y^2+2xy\sqrt{(1+n^2)}-n^2=0$, then that we have

$$x\sqrt{(1+x^2)}+y\sqrt{(1+y^2)}+2nxy-n\sqrt{(1+n^2)}=0,$$

and $$\{x+\sqrt{(1+x^2)}\}\{y+\sqrt{(1+y^2)}\}=n+\sqrt{(1+n^2)},$$

so that the equation of the curve is thus satisfied; wherefore the transcendental curve contains as part of itself the conic $x^2+y^2+2xy\sqrt{(1+n^2)}-n^2=0$.

[As a simple illustration as to how this may happen, take the transcendental curve $y-\sin xy=0$, which it is clear contains as part of itself the line $y=0$].

We have from the equation of the conic

$$\{x+y\sqrt{(1+n^2)}\}^2=n^2(1+y^2),$$

that is $$x+y\sqrt{(1+n^2)}=\pm n\sqrt{(1+y^2)},$$

but considering the radicals as positive, the sign must be taken so that we have simultaneously $x=0$, $y=n$. We have therefore

$$x+y\sqrt{(1+n^2)}=n\sqrt{(1+y^2)},$$

and similarly $$y+x\sqrt{(1+n^2)}=n\sqrt{(1+x^2)},$$

and then, *first*

$$\begin{aligned}n\{x\sqrt{(1+x^2)}+y\sqrt{(1+y^2)}\}&=2xy+(x^2+y^2)\sqrt{(1+n^2)}\\&=2xy+\sqrt{(1+n^2)}\{n^2-2xy\sqrt{(1+n^2)}\}\\&=n^2\{\sqrt{(1+n^2)}-2xy\},\end{aligned}$$

which is the first of the relations in question, and

$$\begin{aligned}&n^2\{x+\sqrt{(1+x^2)}\}\{y+\sqrt{(1+y^2)}\}\\&\quad=n^2xy+nx\{x+y\sqrt{(1+n^2)}\}+ny\{y+x\sqrt{(1+n^2)}\}\\&\qquad+xy+(x^2+y^2)\sqrt{(1+n^2)}+xy(1+n^2)\\&\quad=\{n+\sqrt{(1+n^2)}\}\{x^2+y^2+2xy\sqrt{(1+n^2)}\}\\&\quad=\{n+\sqrt{(1+n^2)}\}n^2,\end{aligned}$$

which is the *second* of the two relations. And the theorem is thus proved.

[The foregoing is the easiest and most obvious solution, but it is interesting to consider the question differently, as follows:

Write $$Q=\frac{\{x+\sqrt{(1+x^2)}\}\{y+\sqrt{(1+y^2)}\}}{n+\sqrt{(1+n^2)}},$$

we have

$$Q\{\sqrt{(n^2+1)}+n\}=\{\sqrt{(1+x^2)}+x\}\{\sqrt{(1+y^2)}+y\}=A+B,$$

$$Q^{-1}\{\sqrt{(n^2+1)}-n\}=\{\sqrt{(1+x^2)}-x\}\{\sqrt{(1+y^2)}-y\}=A-B,$$

if

$$A=\sqrt{(1+x^2)}\sqrt{(1+y^2)}+xy,$$

$$B=x\sqrt{(1+y^2)}+y\sqrt{(1+x^2)},$$

and then

$$\begin{aligned}AB&=x^2y\sqrt{(1+y^2)}+xy^2\sqrt{(1+x^2)}\\&\quad+y(1+x^2)\sqrt{(1+y^2)}+x(1+y^2)\sqrt{(1+x^2)}\\&=x\sqrt{(1+x^2)}+y\sqrt{(1+y^2)}+2xyB,\end{aligned}$$

that is

$$\begin{aligned}&Q^2\{2n^2+1+2n\sqrt{(1+n^2)}\}-\frac{1}{Q^2}\{2n^2+1-2n\sqrt{(1+n^2)}\}\\&\qquad=4\{x\sqrt{(1+x^2)}+y\sqrt{(1+y^2)}\}\\&\qquad\quad+4xy\left[Q\{\sqrt{(1+n^2)}+n\}-\frac{1}{Q}\{\sqrt{(1+n^2)}-n\}\right],\end{aligned}$$

whence

$$\begin{aligned}&4\{x\sqrt{(1+x^2)}+y\sqrt{(1+y^2)}+2nxy-n\sqrt{(1+n^2)}\}\\&=Q^2\{2n^2+1+2n\sqrt{(1+n^2)}\}-\frac{1}{Q^2}\{2n^2+1-2n\sqrt{(1+n^2)}\}\\&\quad+8nxy-4n\sqrt{(1+n^2)}\\&\quad-4xy\left[Q\{\sqrt{(1+n^2)}+n\}-\frac{1}{Q}\{\sqrt{(1+n^2)}-n\}\right]\\&=\left(Q^2-\frac{1}{Q^2}\right)(2n^2+1)+\left(Q^2+\frac{1}{Q^2}-2\right)2n\sqrt{(1+n^2)}\\&\quad-\left(Q-\frac{1}{Q}\right)4xy\sqrt{(1+n^2)}-4\left(Q+\frac{1}{Q}-2\right)nxy\\&=(Q-1)\left\{\frac{(Q+1)(Q^2+1)}{Q^2}(2n^2+1)\right.\\&\quad+\frac{(Q-1)(Q+1)^2}{Q^2}2n\sqrt{(1+n2)}\\&\quad-\frac{4(Q+1)}{Q}xy\sqrt{(1+n^2)}\\&\quad\left.-4\frac{(Q-1)}{Q}nxy\right\}\\&=(Q-1)\,\Omega\text{ suppose,}\end{aligned}$$

that is

$$x\sqrt{(1+x^2)}+y\sqrt{(1+y^2)}+2nxy-n\sqrt{(1+n^2)}=\tfrac{1}{4}(Q-1)\,\Omega.$$

And the integral equation is

$$\tfrac{1}{4}(Q-1)\,\Omega+\log Q=C,$$

which for $C=0$, is satisfied by $Q=1$.

Now starting from

$$Q=\frac{\{x+\sqrt{(1+x^2)}\}\{y+\sqrt{(1+y^2)}\}}{n+\sqrt{(1+n^2)}},$$

we have

$$\sqrt{(1+x^2)}+x=Q\{\sqrt{(1+n^2)}+n\}\{\sqrt{(1+y^2)}-y\}$$

$$\sqrt{(1+x^2)}-x=\frac{1}{Q}\{\sqrt{(1+n^2)}-n\}\{\sqrt{(1+y^2)}+y\},$$

and thence

$$2x=K\sqrt{(1+y^2)}-Ly,$$

if

$$K=Q\{\sqrt{(1+n^2)}+n\}-\frac{1}{Q}\{\sqrt{(1+n^2)}-n\},$$

$$L=Q\{\sqrt{(1+n^2)}+n\}+\frac{1}{Q}\{\sqrt{(1+n^2)}-n\},$$

wherefore $$L^2-K^2=4.$$

Moreover $$(2x+Ly)^2=K^2(1+y^2),$$

that is $$4x^2+(L^2-K^2)y^2+4Lxy=K^2,$$

or what is the same thing

$$x^2+y^2+Lxy=\tfrac{1}{4}(L^2-4),$$

which is the rationalised form of

$$Q=\frac{\{x+\sqrt{(1+x^2)}\}\{y+\sqrt{(1+y^2)}\}}{n+\sqrt{(1+n^2)}}.$$

And if $Q=1$ then $L=2\sqrt{(1+n^2)}$, $\tfrac{1}{4}(L^2-4)=n^2$, so that this equation is $x^2+y^2+2xy\sqrt{(1+n^2)}-n^2=0$; or when $C=0$, the complete integral is satisfied by

$$\frac{\{x+\sqrt{(1+x^2)}\}\{y+\sqrt{(1+y^2)}\}}{n+\sqrt{(1+n^2)}}=1,$$

that is by $$x^2+y^2+2xy\sqrt{(1+n^2)}-n^2=0.$$

We may without difficulty rationalise, and present the result as follows: the equation

$$\left\{2\left(x+\frac{1}{x}\right)+\left(n-\frac{1}{n}\right)\left(y-\frac{1}{y}\right)\right\}\left(1+\frac{1}{x^2}\right)dx$$
$$+\left\{2\left(y+\frac{1}{y}\right)+\left(n-\frac{1}{n}\right)\left(x-\frac{1}{x}\right)\right\}\left(1+\frac{1}{y^2}\right)dy=0,$$

has the complete integral

$$x^2-\frac{1}{x^2}+y^2-\frac{1}{y^2}+\left(x-\frac{1}{x}\right)\left(y-\frac{1}{y}\right)\left(n-\frac{1}{n}\right)-\left(n^2-\frac{1}{n^2}\right)$$
$$=C+4\log\frac{xy}{n},$$

and a particular integral $xy-n=0$: the complete integral is in fact

$$(n-xy)\{-n^3x^2y^2+n^2xy(-x^2-y^2+1)+n(x^2y^2-x^2-y^2)-xy\}$$
$$=x^2y^2n^2\left(C+4\log\frac{xy}{n}\right),$$

satisfied, for $C=0$, by $xy-n=0$].

3. *Write $\alpha=b-c$, $\beta=c-a$, $\gamma=a-b$; then considering the three circles and the three conics*

$$(x-a)^2+y^2=-\beta\gamma,\quad \frac{x^2}{bc}+\frac{y^2}{K+bc}=1,$$
$$(x-b)^2+y^2=-\gamma\alpha,\quad \frac{x^2}{ca}+\frac{y^2}{K+ca}=1,$$
$$(x-c)^2+y^2=-\alpha\beta,\quad \frac{x^2}{ab}+\frac{y^2}{K+ab}=1,$$

where K is arbitrary; it is required to shew that if a variable circle having its centre on one of the conics cuts at right angles the corresponding circle, the envelope is in each of the three cases one and the same bicircular quartic.

Consider the circle $(x-a)^2+y^2=-\beta\gamma$ and the conic $\frac{x^2}{bc}+\frac{y^2}{K+bc}=1$, the coordinates of a point on the conic are $\cos\theta\sqrt{(bc)}$, $\sin\theta\sqrt{(K+bc)}$, where θ is a variable parameter; say for a moment these values are p and q. The equation of the variable circle is

$$(x-p)^2+(y-q)^2=r^2,$$

and in order that this may cut at right angles the circle

$$(x-a)^2+y^2=-\beta\gamma$$

we must have

$$(p-a)^2+q^2=r^2-\beta\gamma,$$

or, substituting for r^2 its value from this equation, the equation of the variable circle is

$$(x-p)^2+(y-q)^2=(a-p)^2+q^2+\beta\gamma,$$

that is $\quad x^2+y^2-a^2-\beta\gamma-2p\,(x-a)-2qy=0,$

viz. this is

$$(x^2+y^2-a^2-\beta\gamma)-2\,(x-a)\,\sqrt{(bc)}\cos\theta-2y\,\sqrt{(K+bc)}\sin\theta=0.$$

Hence taking the envelope in regard to θ, the equation is

$$(x^2+y^2-a^2-\beta\gamma)^2-4\,(x-a)^2bc-4y^2\,(K+bc)=0,$$

that is

$$(x^2+y^2-ab-ac+bc)^2-4\,(x-a)^2bc-4y^2\,(K+bc)=0,$$

or, what is the same thing,

$$(x^2+y^2)^2-2\,(bc+ca+ab)\,(x^2+y^2)-4Ky^2+8abcx$$
$$+b^2c^2+c^2a^2+a^2b^2-2a^2bc-2b^2ca-2c^2ab=0,$$

viz. this equation, being symmetrical in regard to a, b, c, is the same equation as would have been obtained from either of the other conics and the corresponding circle; and from the form of the equation it is clear that the curve is a bicircular quartic.

4. *Shew that the caustic by refraction for parallel rays of a circle, radius* c, *index of refraction* μ, *is the same curve as the caustic by refraction for parallel rays of the concentric circle, radius* $\dfrac{c}{\mu}$, *index of refraction* $\dfrac{1}{\mu}$.

Take as usual $\mu>1$. Imagine the ray AP (fig. 4) parallel to the axis of x, incident at P on the circle radius c, and let the refracted ray after cutting the circle radius $\dfrac{c}{\mu}$, cut it again in Q, and then cut the axis in R. Take ϕ, ϕ' for the angles of incidence and refraction; $\sin\phi=\mu\sin\phi'$.

Moreover in the triangle PQO, we have $\sin Q:\sin P=c:\dfrac{c}{\mu}$; that is $\sin Q=\mu\sin P$, that is $\sin Q=\mu\sin\phi'=\sin\phi$; or

$\angle Q = \phi$. And then in the triangle RQO, $\angle R = \phi - \phi'$, $\angle Q = 180^\circ - \phi$, whence $\angle O = \phi'$, that is $\angle QOR = \phi'$.

Consider now a ray BQ incident at Q and refracted in the direction QR; the index of refraction being $\dfrac{1}{\mu}$, that is the denser medium being on the outside of the small circle. Taking θ, θ' for the angles of incidence and refraction we have $\sin\theta = \dfrac{1}{\mu}\sin\theta'$; but, the refracted ray being by hypothesis QR, we have by what precedes $\theta' = \phi$, hence $\sin\theta = \dfrac{1}{\mu}\sin\phi = \sin\phi'$, that is $\theta = \phi'$, or $\angle BQO = \angle QOR$, that is the incident ray BQ is parallel to the axis of x. We have thus two pencils of rays each parallel to the axis, such that for any ray AP of the first pencil there is a corresponding ray BQ of the second pencil, the rays AP and BQ each giving rise to the same refracted ray PQR; hence the two pencils have the same caustic.

[It is proper to remark that for the ray BQ it has been assumed that the refraction takes place not at Q' where it first meets the small circle, but at Q; if we consider the refraction at Q', then the index of refraction is still to be $= \dfrac{1}{\mu}$, that is the denser medium must now be *inside* the small circle; the refracted ray is in the direction $R'Q'$ situate symmetrically with RQ on the opposite side of the axis of y; and it would at first sight appear that the caustic was a curve equal and similar to the original caustic, but situate on the opposite side of the axis of y. But geometrically the complete caustic consists of two equal and similar portions situate on opposite sides of the axis of y; so that we really obtain, not an equal and opposite caustic, but in each case one and the same caustic.

I originally obtained the theorem in a different manner; viz. the equation for the caustic for the first pencil of rays was found to be

$$(1-\mu^2)\,\frac{x}{c} = \left\{1 - \mu^{\frac{4}{3}}\left(\frac{y}{c}\right)^{\frac{2}{3}}\right\}^{\frac{3}{2}} + \mu\left\{1 - \mu^{-\frac{2}{3}}\left(\frac{y}{c}\right)^{\frac{2}{3}}\right\}^{\frac{3}{2}},$$

which equation (as is easily seen) remains unaltered when c, μ are changed into $\dfrac{c}{\mu}, \dfrac{1}{\mu}$ respectively.—See my "Memoir on Caustics," *Phil. Trans.*, t. 147 (1857), p. 285.]

5. *Given at each point of space the direction-cosines* (α, β, γ) *of a line through that point: it is required to find the conditions in order that the lines may be not a triple but a double system.*

For any given point P the values of the quantities α, β, γ which determine the direction of the line through that point are given as functions of the coordinates (x, y, z) of the point P. Hence passing from a point P to a consecutive point P' *on the line*, the coordinates of P' will be $x+\rho\alpha$, $y+\rho\beta$, $z+\rho\gamma$; and the values of α, β, γ for the point P' will be

$$\alpha+\rho\left(\alpha\frac{d\alpha}{dx}+\beta\frac{d\alpha}{dy}+\gamma\frac{d\alpha}{dz}\right),$$

$$\beta+\rho\left(\alpha\frac{d\beta}{dx}+\beta\frac{d\beta}{dy}+\gamma\frac{d\beta}{dz}\right),$$

$$\gamma+\rho\left(\alpha\frac{d\gamma}{dx}+\beta\frac{d\gamma}{dy}+\gamma\frac{d\gamma}{dz}\right).$$

But if the lines form a double system, we must have the same line for the point P, and for any other point P' on the line; and in particular the same line for the point P, and for the consecutive point P'. Hence as conditions for the double system we obtain

$$\alpha\frac{d\alpha}{dx}+\beta\frac{d\alpha}{dy}+\gamma\frac{d\alpha}{dz}=0,$$

$$\alpha\frac{d\beta}{dx}+\beta\frac{d\beta}{dy}+\gamma\frac{d\beta}{dz}=0,$$

$$\alpha\frac{d\gamma}{dx}+\beta\frac{d\gamma}{dy}+\gamma\frac{d\gamma}{dz}=0.$$

But in virtue of the relation $\alpha^2+\beta^2+\gamma^2=1$ we have

$$\alpha\frac{d\alpha}{dx}+\beta\frac{d\beta}{dx}+\gamma\frac{d\gamma}{dx}=0,$$

$$\alpha\frac{d\alpha}{dy}+\beta\frac{d\beta}{dy}+\gamma\frac{d\gamma}{dy}=0,$$

$$\alpha\frac{d\alpha}{dz}+\beta\frac{d\beta}{dz}+\gamma\frac{d\gamma}{dz}=0.$$

Hence subtracting the corresponding equations we have

three equations, which are at once seen to be equivalent to the two equations

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy}:\frac{d\gamma}{dx}-\frac{d\alpha}{dz}:\frac{d\alpha}{dy}-\frac{d\beta}{dx}=\alpha:\beta:\gamma,$$

equations which must be satisfied identically, whatever are the values of (x, y, z). The equations have been obtained as necessary conditions; they are, in fact, the sufficient conditions for a double system; for the line being unaltered in passing from P to P', it remains unaltered when we pass to the following point P'', and so on; that is for the passage to any point Q whatever on the line.

COR. If the equation $\alpha dx+\beta dy+\gamma dz=0$ be integrable by a factor, it must be integrable *per se*: in fact, the condition that it may be integrable by a factor is

$$\alpha\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+\beta\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)+\gamma\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)=0.$$

But we have

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy}=k\alpha, \text{ \&c.},$$

and the equation thus becomes

$$k(\alpha^2+\beta^2+\gamma^2)=0,$$

that is $k=0$, and therefore

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy}=0,\quad \frac{d\gamma}{dx}-\frac{d\alpha}{dz}=0,\quad \frac{d\alpha}{dy}-\frac{d\beta}{dx}=0.$$

Hence, also, if the lines cut at right angles a surface, we must have $\alpha dx+\beta dy+\gamma dz$ a complete differential.

The foregoing theory is given in Sir W. R. Hamilton's "Memoir on Ray-Systems."

6. *If $X=0$, $Y=0$, $Z=0$, $W=0$ are four given conics in the same plane and having a common point: shew that in the system of conics $aX+bY+cZ+dW=0$ there are in general four (improper) conics the equations of which may be taken to be $x^2=0$, $y^2=0$, $xz=0$, $yz=0$.*

Taking the conics to pass through the point $x=0$, $y=0$; their equations will be of the form

$$\begin{aligned}X&=a_1x^2+2h_1xy+b_1y^2+2f_1yz+2g_1zx=0,\\ Y&=a_2x^2+2h_2xy+b_2y^2+2f_2yz+2g_2zx=0,\\ Z&=a_3x^2+2h_3xy+b_3y^2+2f_3yz+2g_3zx=0,\\ W&=a_4x^2+2h_4xy+b\,y^2+2f_4yz+2g_4zx=0.\end{aligned}$$

Now multiplying by the indeterminate quantities α, β, γ, δ, the three ratios $\alpha : \beta : \gamma : \delta$ may be determined so that the terms in yz, zx shall vanish, and the terms in x^2, xy, y^2 be a perfect square: we thus arrive at a quadric equation for any one of the ratios, say $\alpha : \beta$, the remaining ratios being then linearly determined; viz. there are two sets of values of α, β, γ, δ: and changing the coordinates (x, y), the two resulting forms may be represented by $x^2 = 0$, $y^2 = 0$.

And it is clear that we thus have in the system of conics $\alpha X + \beta Y + \gamma Z + \delta W = 0$, four conics the equations of which may be represented by

$$\begin{aligned} X' &= x^2 &&= 0, \\ Y' &= y^2 &&= 0, \\ Z' &= h_3 xy + f_3 yz + g_3 zx &&= 0, \\ W' &= h_4 xy + f_4 yz + g_4 zx &&= 0, \end{aligned}$$

where of course the coefficients f, g, h have new values.

We may then form the equations

$$\alpha X' + f_4 Z' - f_3 W' = x\{\alpha x + (f_4 h_3 - f_3 h_4)\, y + (f_4 g_3 - f_3 g_4)\, z\},$$

$$\beta Y' - g_4 Z' + g_3 W' = y\{(g_3 h_4 - g_4 h_3)\, x + \beta y + (f_4 g_3 - f_3 g_4)\, z\},$$

so that by writing $\alpha = g_3 h_4 - g_4 h_3$ and $\beta = f_4 h_3 - f_3 h_4$, the terms in $\{\ \}$ will be one and the same linear function of (x, y, z); that is changing the z so as to denote the linear function in question by z, we have as conics of the series $xz = 0$, and $yz = 0$, that is we have in the series the four conics $x^2 = 0$, $y^2 = 0$, $xz = 0$, $yz = 0$; whence also any other conic of the series, and consequently each of the original four conics, may be represented by an equation of the form

$$ax^2 + by^2 + 2fyz + 2gzx = 0.$$

(To be continued).

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

At a Meeting of this Society, held Thursday, June 8th., at 8 P.M., W. Spottiswoode, Esq., F.R.S., President, in the chair, Mr. W. Chadwick, M.A., Fellow of Corpus Christi College, and Mr. J. Griffiths, M. A., Fellow of Jesus College, Oxford, were elected Members.

Prof. Cayley gave an account of Plücker's Models of certain Quartic Surfaces. These wooden models, presented to the Society by Dr. Hirst in December, 1866, are 14 in number, and were constructed under the direction of the late Professor Plücker in illustration of the theory developed in his posthumous work "Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raum Elemente," Leipzig, 1869. All of them, Prof. Cayley believes, represent Equatorial surfaces. Models 1 to 8 representing cases of the 78 forms of equatorial surfaces "deren Breiten-curven eine feste axen-richtung besitzen," vol 2, pp. 352—363; the remaining models, 9 to 14, he has not completely identified.

The paper went into the theory only so far as is required for the explanation of the models.

In a "complex" or triply infinite system of lines there is in any plane whatever a single infinite system of lines enveloping a curve; and if we attend only to the curves, the planes of which pass through a given fixed line, the locus of these curves is a "complex surface."

The models in the Society's collection are numbered consecutively from 1 to 14, Prof. Cayley gave the following as his identification: model 1 is Plücker's 13, 2 is 9, 3 is 40, 4 is 34, 5 is 2, 6 is 3, 7 is 4, and 8 is Plücker's 32.

Mr. S. Roberts, M.A., then gave in detail an account of his paper on "the motion of a plane under certain conditions."

Discussions took place on both communications. Prof. Henrici, V.P., exhibited cardboard models of 2 ellipsoids, a hyperboloid of one sheet, and of an elliptic paraboloid, also stereograms of the models of surfaces exhibited at former Meetings of the Society.

A brief discussion followed on the subject of Prof. Cayley's communication (p 31), but it was determined that no definite action should be taken in the matter at present.

Sixteen presents of books and pamphlets were made to the Library, including "Sulle trasformazioni razionali nello spazio" nota 1. del Prof. L. Cremona, from the author.

R. TUCKER. M A., *Hon. Sec.*

Cambridge Philosophical Society.

At a Meeting of this Society, held Monday, May 29th, Professor Cayley, F.R.S., President, in the chair, Dr. M. Foster, M.A., Trinity, was elected a Fellow; and the following were elected Honorary Members: Prof. Sir B. C. Brodie, M.A., F.R.S.; W. B. Carpenter, F.R.S.; A. R. Clarke, Capt. R.E., F.R.S.; Prof. T. Huxley, M.D., F.R.S.; Prof. B. Price, M.A., F.R.S.; W. Spottiswoode, M.A., F.R.S.; Professor F. W. A. Argelander (Bonn); A. Clebsch (Göttingen); A. O. des Cloiseaux (Paris); H. Helmholtz (Berlin); F. Wöhler (Göttingen).

The following communications were made to the Society: "On an Illustration of the Empirical Theory of Vision," by Mr. Coutts Trotter, in which the author gave an account of some experiments he had made with regard to his own eyes. With one eye he was short-sighted, the other had the ordinary range of vision; but on using spectacles, so as to make both eyes have the same focal length, and thus become perfect instruments, he did not see so accurately as when one was an imperfect instrument. "On a Table of the Logarithms of the first 250 Bernoulli's Numbers," by Mr. J. W. L. Glaisher; the logarithms, which were given to 10 places, were most of them calculated from the formula

$$\log B_n = \log 1 + \log 2 \ldots + \log (2n) - 2n \log (2\pi) + \log 2$$
$$+ \mu \left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \ldots\right),$$

μ being the modulus $\cdot 434294 \ldots$.

ON A PROPERTY IN THE THEORY OF CONFOCAL QUADRICS.

By *R. Townsend, M.A., F.R.S.*

THE following property, of which it is proposed to give a geometrical demonstration, appeared in a late number of *Terquem*.

The edge of regression of the imaginary developable circumscribed to a system of confocal quadrics projects orthogonally upon any principal plane of the system into the evolute of the focal conic in that plane.

As all quadrics of the same confocal system have a common circumscribing developable, to establish this property generally it will be sufficient to prove it for the developable circumscribed to a focal conic and to any surface of the system, which may be easily done as follows. Conceiving a plane to revolve round the tangent at any point P of the conic, its pole Q with respect to the surface, by a known property of confocal quadrics, lies in every position on the normal to itself at the point P, and therefore does so in the particular [and of course imaginary] position in which it touches the surface, in which case the pole Q becomes the point of contact, and the line PQ a generator of the developable in question, which generator consequently lies in the normal plane to the conic at P, and therefore projects orthogonally upon the plane of the conic into the normal in its plane at P; and, the same being true of every generator of the developable with respect to its point of intersection with the conic, therefore &c.

The above property transforms homographically into the following, of which it is manifestly a particular case.

For a system of quadrics having a common circumscribing developable, and therefore a common self-reciprocal tetrahedron, the edge of repression of the developable connects with any vertex of the tetrahedron by a cone, which intersects with the opposite face, not of course generally in the evolute of the infinitely flat quadric of the system lying in that plane, but in the envelope of the system of lines harmonically conjugate to its system of tangents with respect to the triads of angles in involution determined at their points of contact by the three

pairs of opposite vertices of the quadrilateral in which the developable intersects with the same plane.

With obvious modifications, the demonstration above given for the particular case holds for the more general property also.

Trinity College, Dublin,
July 8*th*, 1871.

NOTE ON CURVATURE.

By *E. C. Constable, M.A.*

To find the central chord of curvature at any point of a conic.

Let QVQ' (fig. 5) be any double ordinate to the diameter PP', and let the circle circumscribing the triangle PQQ'* cut PP' in H.

Then $\qquad QV^2 = PV.VH$ (Euc. III. 35).

Therefore $\qquad PV.VH : PV.VP' :: CD^2 : CP^2,$

or $\qquad VH : VP' :: CD^2 : CP^2.$

Hence ultimately when Q, V, Q' coincide with P,

$$PH : 2CP :: CD^2 : CP^2,$$

or $$PH.CP = 2CD^2,$$

where PH is the central chord of curvature at P.

In the case of the parabola,

$$PV.VH = 4SP.PV, \text{ or } VH = 4SP.$$

Hence ultimately $\qquad PH = 4SP,$

where PH is the chord of curvature parallel to the axis.

Murath, 1871.

---

* Professor Townsend—see Salmon's *Conics*—uses a like construction in determining the *focal* chord of curvature.—*Ed.*

OSCILLATIONS OF A HEAVY STRING.

By *E. J. Routh, M.A.*

CONTENTS.

1. IN the seventh volume of the *Journal Polytechnique*, Poisson discusses the oscillations of a heavy chain suspended by one extremity, the extent of the vibration of every particle of the chain on each side of the vertical being extremely small. If the fixed point be taken as the origin and the vertical as the axis of x, and l be the length of the chain, the equation of motion is easily seen to be

$$\frac{d^2y}{dt^2} = g(l-x)\frac{d^2y}{dx^2} - g\frac{dy}{dx} \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Putting
$$s = \sqrt{(l-x)} + \tfrac{1}{2}\sqrt{(g)}\,t,$$
$$s' = \sqrt{(l-x)} - \tfrac{1}{2}\sqrt{(g)}\,t,$$
$$\frac{y'}{\sqrt[4]{(l-x)}} = y.$$

Poisson reduces the equation to the form

$$\frac{d^2y'}{ds\,ds'} = -\frac{1}{4}\,\frac{y'}{(s+s')^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The integral of this equation he finds to be

$$y' = \int_a^s Hf(h)\,dh + \int_{a'}^{s'} H'F(h)\,dh \quad \ldots\ldots\ldots (3).$$

where $f(h)$ and $F(h)$ are two arbitrary functions of h: and H, H' stand for two infinite series, and are functions of h, s, and s'.

He now supposes that the primitive disturbance extends throughout a space determined by $x = c$ to $x = c + \alpha$, and that the remaining part of the chain is at rest in the axis of x. After a rather long discussion of the forms of the arbitrary functions, he finds that there will be two oscillations, one of which travels down the chain with a uniform retardation equal to $\tfrac{1}{2}g$, the initial velocity being $\sqrt{\{g(l-c-\alpha)\}}$, and the other up the chain with a uniform acceleration also equal to $\tfrac{1}{2}g$, the initial velocity being $\sqrt{\{g(l-c)\}}$.

Poisson also remarks that these results are still correct, if we suppose a heavy particle to be attached to the lowest point of the chain.

2. The following general considerations will show what might have suggested these transformations. We may then apply these considerations to discover the corresponding transformations for a heterogeneous chain of any kind suspended by both extremities at any two points in space. It will be found that we may determine the motion more or less completely in several cases.

Let us suppose that the displacements of the particles forming any finite portion of the chain during a finite time, are represented by $y = \phi(x, t)$, where ϕ is a continuous function of x and t. Let P be a geometrical point within this portion of the chain which moves so that the particle-velocity at P, *i.e.* $\frac{dy}{dt}$ is always equal to some constant quantity M. Let v be the velocity with which P moves, then we have following in our mind the motion of P,

$$\frac{d^2y}{dt^2} + \frac{d^2y}{dx\,dt}\,v = 0.$$

Let Q be a point also within the portion, such that the tangent to the chain at Q makes with the vertical an angle whose tangent, *i.e.* $\frac{dy}{dx}$, is $\frac{N}{l-x}$, where N is some constant quantity.

Let v' be the velocity with which Q moves, then

$$(l-x)\frac{d^2y}{dx\,dt} + \frac{d}{dx}\left\{(l-x)\frac{dy}{dx}\right\}v' = 0.$$

The differential equation of motion may be written in the form

$$\frac{d^2y}{dt^2} = g\frac{d}{dx}\left\{(l-x)\frac{dy}{dx}\right\}.$$

From these equations we easily deduce that if P and Q coincide at any instant

$$vv' = g(l-x).$$

This reasoning requires that all the second differential coefficients should be finite, and that y is a continuous function of x and t. It would not apply to any point P, if the discontinuous extremities of two waves were passing over P in opposite directions. But the consideration of these exceptions is unnecessary for our present purpose.

3. Let AB be a disturbed portion of the chain travelling in the direction AB on a chain otherwise in equilibrium.

At the confines of the disturbance the two portions of the string must not make a finite angle with each other. If they did, an element of the string would be acted on by a finite moving force, which is the resultant of the two finite tensions at its extremities. In such a case, the disturbance would instantly extend itself further along the chain and take up some new form. Supposing we exclude any such case as this, we must have, as long as the motion is finite, both $\frac{dy}{dt}=0$, and $\frac{dy}{dx}=0$, at both the upper and lower extremity of the disturbance. If then P be a point at which $\frac{dy}{dt}=0$, and Q a point at which $\frac{dy}{dx}=0$, P and Q may be considered as taken just within the boundary of the wave; P and Q will therefore each travel with the velocity of that boundary. Hence putting $v=v'$, we find for the velocity of either boundary

$$v=\sqrt{\{g\,(l-x)\}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where x is the abscissa of that boundary. *This is exactly the result arrived at by Poisson.*

4. The velocity of a boundary of a small disturbance being supposed to be given by

$$\frac{dx}{dt}=\pm\sqrt{\{g\,(l-x)\}},$$

let us represent the motion of a point P by this equation.

Integrating, we get

$$\left.\begin{matrix}\sqrt{(l-x)}+\frac{1}{2}\sqrt{(g)}\,t=s\\ \sqrt{(l-x)}-\frac{1}{2}\sqrt{(g)}\,t=s'\end{matrix}\right\},$$

where s and s' are two arbitrary constants, such that when s' is constant we make P travel up the chain with an acceleration $\frac{1}{2}g$, and when s is constant we make P travel down with the same acceleration. If we wish to determine the changes in the form of the wave in the neighbourhood of P as it travels along the string, there is an obvious advantage in taking s and s' as independent variables, and thus we are led to the very transformation used by Poisson.

5. If the chain be heterogeneous the fundamental equation of motion takes a slightly different form. Let D be the density at a distance x below the point of suspension, then we easily find

$$\frac{d^2y}{dt^2}=g\,\frac{\int_x^l Ddx}{D}\,\frac{d^2y}{dx^2}-g\,\frac{dy}{dx}\ldots\ldots\ldots\ldots\ldots(5),$$

where $\int_x^l Ddx$ is the whole weight below the point of the chain under consideration. If a small disturbance be given extending over any small part of the chain, we find, by the same reasoning as before, that the velocity of either boundary is given by

$$v^2 = g\frac{\int_x^l Ddx}{D} \quad \dots\dots\dots\dots\dots\dots\dots (6),$$

where x is the abscissa of that boundary.

If we put $t+\int\frac{dx}{v}=s$, $-t+\int\frac{dx}{v}=s'$, and change the independent variables to s and s', the equation becomes

$$\frac{d^2y}{dsds'} - \tfrac{1}{4}\left(\frac{dv}{dx}+\frac{g}{v}\right)\left(\frac{dy}{ds}+\frac{dy}{ds'}\right)=0 \quad \dots\dots\dots (7).$$

Putting $y=\frac{y'}{z}$ and choosing z, so that $\frac{v}{z}\frac{dz}{dx}=-\frac{1}{2}\left(\frac{dv}{dx}+\frac{g}{v}\right)$, this equation reduces to

$$\frac{d^2y'}{dsds'}=f(s+s')\,y \quad \dots\dots\dots\dots\dots\dots\dots (8).$$

This is of the same form as that to which Poisson reduces his equation and an integral may be found similar to that given in (3) of Art. 1.

6. There are some cases in which this equation may be completely solved. Let us suppose the law of density to be given by

$$D=\frac{A}{\sqrt{(l+l'-x)}},$$

where l as before is the length of the chain and l' any constant, and let a weight $W=2Ag\sqrt{(l')}$ be fastened at the lower end of the chain, the equation (5) takes the form

$$\frac{d^2y}{dt^2}=g\left\{2(l+l'-x)\frac{d^2y}{dx^2}-\frac{dy}{dx}\right\} \quad \dots\dots\dots (9).$$

The solution is easily seen from the tranformation in equation (7) to be

$$y=f\left\{\sqrt{(l+l'-x)}-\sqrt{\left(\frac{g}{2}\right)}t\right\}+F\left\{\sqrt{(l+l'-x)}+\sqrt{\left(\frac{g}{2}\right)}t\right\}\dots(10),$$

where f and F are two arbitrary functions. When a small

disturbance travels up or down this chain the velocity of either boundary is that due to the length of a chain whose weight is equal to the total weight below that boundary. More generally, if any point P move so that its velocity is that due to the length of a chain whose weight is equal to the weight below P, then the ordinate y of the disturbance at P is always the same.

7. The simplicity of the form of y when the density follows this law enables us to discuss the motion as completely as we please. Thus, if the whole chain and weight be set in motion so as to sound a musical note, we have the disturbance represented by any number of terms of the form

$$y = A \sin \kappa \left\{\sqrt{(l + l' - x)} - \sqrt{\left(\frac{g}{2}\right)}\, t + \alpha\right\}$$
$$+ B \sin \kappa \left\{\sqrt{(l + l' - x)} + \sqrt{\left(\frac{g}{2}\right)}\, t + \beta\right\}.$$

To find the possible values of κ, we have the conditions (1) when $x = 0$, then $y = 0$ for all values of t, (2) when $x = l$, then y must satisfy the equation of motion of the weight, viz.

$$\frac{W}{g}\frac{d^2y}{dt^2} = -W\frac{dy}{dx}.$$

These easily lead to the equation

$$\kappa \sqrt{(l')} \tan \kappa \{\sqrt{(l + l')} - \sqrt{(l')}\} = 1 \text{.........}(11).$$

8. If a heterogeneous chain be suspended by its two extremities A and B, the equations of motion may be treated in a similar manner. Let the axis of x be horizontal, that of y vertical. Let C be any fixed point on the chain when hanging in equilibrium, xy the coordinates of any point P determined by $CP = s$. Let T be the tension at P, $mgds$ the weight of an element ds situated at P. The equations of equilibrium are

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) = 0, \quad \frac{d}{ds}\left(T\frac{dy}{ds}\right) - mg = 0.$$

Let α be the angle the tangent at P makes with the axis of x, then we easily find

$$T = \frac{Ag}{\cos \alpha}, \quad m = A\frac{d \tan \alpha}{ds} \text{ ....}(12),$$

where A is an undetermined constant.

When the chain is in motion, let $(x+\xi, y+\eta)$ be the coordinates of the position of the particle P at the time t, and let the tension at that point be $T' = T + U$. The equations of motion will be

$$\frac{d^2\xi}{dt^2} = \frac{1}{m}\frac{d}{ds}\left\{T'\left(\frac{dx}{ds} + \frac{d\xi}{ds}\right)\right\},$$

$$\frac{d^2\eta}{dt^2} = \frac{1}{m}\frac{d}{ds}\left\{T'\left(\frac{dy}{ds} + \frac{d\eta}{ds}\right)\right\} - g,$$

which, by subtracting the equations of equilibrium, reduce to

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} &= \frac{1}{m}\frac{d}{ds}\left(T\frac{d\xi}{ds} + U\frac{dx}{ds}\right)\\ \frac{d^2\eta}{dt^2} &= \frac{1}{m}\frac{d}{ds}\left(T\frac{d\eta}{ds} + U\frac{dy}{ds}\right)\end{aligned}\right\}\ldots\ldots\ldots\ldots(13),$$

when the squares of small quantities are neglected. Also as in the author's *Rigid Dynamics*, p. 481, we have the geometrical equation

$$\frac{d\xi}{ds} = -\frac{dy}{dx}\frac{d\eta}{ds}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

9. If we suppose a small disturbance to travel along this chain, so that there is no abrupt change of direction of the chain at the boundaries of the wave, we must have at those points $\frac{d\xi}{ds} = 0$, $\frac{d\eta}{ds} = 0$, $\frac{d\xi}{dt} = 0$, $\frac{d\eta}{dt} = 0$, and $U = 0$. Let v be the velocity with which one boundary of this wave travels along the chain, then, following that boundary in our mind, we have

$$\frac{d^2\xi}{dt^2} + v\frac{d^2\xi}{ds\,dt} = 0, \quad \frac{d^2\xi}{dt\,ds} + v\frac{d^2\xi}{ds^2} = 0,$$

and therefore
$$\frac{d^2\xi}{dt^2} = v^2\frac{d^2\xi}{ds^2},$$

with a similar equation for η. Thus the dynamical equations become

$$\left.\begin{aligned}\left(v^2 - \frac{T}{m}\right)\frac{d^2\xi}{ds^2} &= \frac{dU}{ds}\frac{dx}{ds}\\ \left(v^2 - \frac{T}{m}\right)\frac{d^2\eta}{ds^2} &= \frac{dU}{ds}\frac{dy}{ds}\end{aligned}\right\},$$

and the geometrical equation becomes

$$\frac{d^2\xi}{ds^2}\frac{dx}{ds} = -\frac{d^2\eta}{ds^2}\frac{dy}{ds}.$$

From these we easily get $v^2 = \frac{T}{m}$. Substituting for T and m their values, we have if ρ be the radius of curvature at P,

$$v = \sqrt{(g\rho\cos\alpha)} \text{ .... } (15),$$

so that the velocity of either boundary of the wave is that due to one quarter of the vertical chord of curvature at that point.

10. If we wish to solve the equations of motion of a heterogeneous chain, it will be convenient to express the unknown quantities ξ, η, U, in terms of some *one* function ϕ. Let $\alpha + \phi$ be the angle the tangent at P makes with the horizon. Then, as in *Rigid Dynamics*, p. 480, we have

$$\frac{d\xi}{ds} = -\sin\alpha\phi,\ \frac{d\eta}{ds} = \cos\alpha\phi \text{ .... } (16).$$

Let

$$\left.\begin{aligned} X &= \xi\cos\alpha + \eta\sin\alpha \\ Y &= -\xi\sin\alpha + \eta\cos\alpha \end{aligned}\right\} \text{ .... } (17),$$

so that X and Y are the displacements of P parallel to the tangent and normal to the chain in its position of equilibrium. The equations of motion will then be found to give

$$\left.\begin{aligned} \frac{d^2X}{dt^2} &= -g\cos\alpha\phi + \frac{\cos^2\alpha}{A}\frac{dU}{d\alpha} \\ \frac{d^2Y}{dt^2} &= g\sin\alpha\phi + g\cos\alpha\frac{d\phi}{d\alpha} + \frac{\cos^2\alpha}{A}U \end{aligned}\right\} \text{...} (18),$$

$$\left.\begin{aligned} Y &= \frac{dX}{d\alpha} \\ \frac{d^2X}{d\alpha^2} + X &= \phi\frac{ds}{d\alpha} \end{aligned}\right\} \text{ .... } (19).$$

Treating these equations in the same way as the corresponding equations in *Rigid Dynamics*, p. 477, we get

$$\rho\cos\alpha\frac{d^2\phi}{dt^2} - 2\frac{d^2\xi}{dt^2} = g\left(\cos^2\alpha\frac{d^2\phi}{d\alpha^2} + 2\sin\alpha\cos\alpha\frac{d\phi}{d\alpha} + 2\phi\right) \text{...} (20),$$

and thence by eliminating ξ

$$\frac{d^2\phi}{dt^2} = g\frac{\cos\alpha}{\rho}\left\{\frac{d^2\phi}{d\alpha^2} + 4\phi + f(t)\right\} \text{ .... } (21),$$

where $f(t)$ is an arbitrary function of t.

11. This equation may be put under another form. The velocity of either boundary of a small wave is given by $\frac{ds}{dt} = \sqrt{(g\rho \cos\alpha)}$ or which is the same thing $\frac{d\alpha}{dt} = \sqrt{\left(\frac{g \cos\alpha}{\rho}\right)}$. If we put $\mu^2 = \frac{g \cos\alpha}{\rho}$ the integral of this equation is

$$t - \int \frac{d\alpha}{\mu} = \text{constant}.$$

Let us then change the indepedent variables from t and α to p and q where

$$t + \int \frac{d\alpha}{\mu} = p, \quad -t + \int \frac{d\alpha}{\mu} = q.$$

The equation then becomes

$$\frac{d^2\phi}{dpdq} - \tfrac{1}{4}\left(\frac{d\phi}{dp} + \frac{d\phi}{dq}\right)\frac{d\mu}{d\alpha} + \mu^2\left(\phi + \frac{f(t)}{4}\right) = 0 \ldots (22).$$

If we put $\frac{\phi}{\sqrt{\mu}} = \phi'$ and $\mu = \mu'^2$, the equation takes the simple form

$$\frac{d^2\phi'}{dpdq} + \frac{\mu'^3}{4}\left(\frac{d^2\mu'}{d\alpha^2} + 4\mu'\right)\phi' = -\mu'^3\frac{f(t)}{4} \ldots\ldots (23),$$

where $\mu'^4 = \frac{g \cos\alpha}{\rho}$ and $2\int \frac{d\alpha}{\mu} = p + q$. The coefficient of ϕ' just as in Poisson's equation (2) is therefore some function of $p+q$, the form of the function in this case depending on the law of density of the chain.

When this equation has been integrated, we have ϕ expressed in terms of two arbitrary functions of determinate combinations of α and t. We may then find ξ and η from equations (16) with two arbitrary functions of t. These arbitrary functions may be determined from the known initial motion of the string, and the known motions of the points A, B by which the string is suspended. As the final equation (21) has been obtained from (20) by differentiation and integration, the function $f(t)$ thus introduced may be found by substituting in equation (20). In many cases the circumstances of the problem will enable us to determine at once the value of $f(t)$. Thus, suppose the string when in equilibrium to be symmetrical about a vertical line, say the axis of y, and let the points of support be fixed in the same horizontal line. Then if the initial motion be also symmetrical about the axes of y, the whole subsequent motion will be symmetrical. Thus ϕ must be a function of α, containing

when expanded only odd powers of α. Substituting such a series in equation (21), we see that $f(t)$ must be zero.

12. The tension of the string at any point may be found from the second of equations (18). But another expression is

$$\cos\alpha \,.\, T = \frac{Ag}{2}\int\{4\phi + f(t)\}\,d\alpha \ldots\ldots\ldots\ldots (24).$$

13. An approximate solution of equation (23) may be found in the following manner: let us suppose a small disturbance given to any point of a very long string, there will generally be two waves produced which travel in opposite directions. If we follow only one of these waves, then after a time one of the independent variables p and q will become large. Let us consider the wave in which p is large. The coefficients may in certain cases be expanded in descending powers of p. Retaining as an approximation only the highest powers which occur, we have

$$\frac{d^2\phi'}{dpdq} + Mp^r\phi' = Np^{r'}.$$

The integral of this equation is

$$\phi = \mu' \frac{N}{M} p^{r'-r} + \mu'\Sigma A \sin\left\{\kappa q - \frac{Mp^{r+1}}{\kappa(r+1)}\right\},$$

where A and κ may have any values whatever.

14. There are several cases in which the equation to find the small motions of a chain may be more or less completely integrated. One of the most interesting of these is that in which the chain hangs in equilibrium in the *form of a cycloid.* In this case we have, if a be the radius of the generating circle, $\rho = 4a \cos\alpha$. The density of the chain at any point is given by $m = \dfrac{A}{a\cos^3\alpha}$, so that all the lower part of the chain is of nearly uniform density, but the density increases rapidly higher up the chain and is infinite at the cusp.

The equation to find the oscillations now takes the simple form

$$\frac{d^2\phi}{dt^2} = \frac{g}{4a}\left\{\frac{d^2\phi}{d\alpha^2} + 4\phi + f(t)\right\},$$

in which all the coefficients are constants.

There are two cases of motion to be discussed, (1) when the chain swings up and down, and (2) when it swings from side to side.

CASE I. Let us suppose the motion of the chain to be symmetrical about the axis of the cycloid, and the points of support equally distant from the vertex, so that $f(t) = 0$. In this case the middle point of the chain oscillates up and down without any lateral motion. To find the nature and time of a small oscillation, we put

$$\phi = R \sin \kappa t,$$

where R is a function of α only. Substituting, we have

$$\frac{d^2 R}{d\alpha^2} + 4\left(1 + \frac{a\kappa^2}{g}\right) R = 0;$$

therefore $$R = L \sin 2 \sqrt{\left(1 + \frac{a\kappa^2}{g}\right)}\, \alpha,$$

where L is an arbitrary constant, the other constant being determined by the consideration that the motion is symmetrical about the axis of y. For the sake of brevity, put $\lambda = 2\sqrt{\left(1 + \frac{a\kappa^2}{g}\right)}$. Substituting in (16), we find

$$\xi = L \frac{2a}{\lambda^2 - 4} \{\lambda \cos\lambda\alpha \sin 2\alpha - 2 \sin\lambda\alpha \cos 2\alpha\} \sin \kappa t,$$

$$\eta = \left[-L \frac{2a}{\lambda^2 - 4} \{\lambda \cos\lambda\alpha \cos 2\alpha + 2 \sin\lambda\alpha \sin 2\alpha\} - L \frac{2a}{\lambda} \cos\lambda\alpha + H\right] \sin \kappa t,$$

where H is a constant depending on the position of the points of support. Let the length of the chain be $2l$, then at either end $\sin\alpha_0 = \frac{l}{4a}$. At both extremities we must have $\xi = 0$, $\eta = 0$. All these four conditions can be satisfied if

$$\frac{\tan\lambda\alpha_0}{\lambda} = \frac{\tan 2\alpha_0}{2}.$$

This equation therefore determines the possible times of symmetrical vibration of a heterogeneous chain hanging in the form of a cycloid.

If α be not very large, the oscillations are nearly the same as those of a uniform chain. In this case we have $\tan\lambda\alpha_0 = \lambda\alpha_0$ nearly. The least value of $\lambda\alpha$ which can be

taken is a little less than $\frac{3\pi}{2}$. Hence λ is great, and therefore $\kappa = \sqrt{\left(\frac{g}{4a}\right)}\,\lambda$ nearly.

CASE II. Let us now suppose the chain swinging from side to side, so that the middle point has only a lateral motion without any perceptible vertical motion. To find the time of oscillation, we put

$$\phi = R \sin \kappa t,$$

where R is a function of α only. Substituting in (21), we see that $f(t) = h \sin \kappa t$, where h is an arbitrary constant. The equation to find R becomes

$$\frac{d^2R}{d\alpha^2} + 4\left(1 + \frac{a\kappa^2}{g}\right) R = -h.$$

If we put $\lambda^2 = 4\left(1 + \frac{a\kappa^2}{g}\right)$ as before, we find

$$R = -\frac{h}{\lambda^2} + L \sin(\lambda\alpha + M).$$

Thence $$\frac{\xi}{\sin \kappa t} = \frac{h' - ha \cos 2\alpha}{\lambda^2}$$

$$+ L \frac{2a}{\lambda^2 - 4} \{\lambda \cos(\lambda\alpha + M) \sin 2\alpha - 2 \sin(\lambda\alpha + M) \cos 2\alpha\},$$

where h' is an arbitrary constant introduced on integration. Substituting in equation (20), we find

$$h' = -h\left(a + \frac{g}{\kappa^2}\right),$$

also, we have in the same way

$$\frac{\eta}{\sin \kappa t} = -\frac{ha}{\lambda^2}(2\alpha + \sin 2\alpha)$$

$$- L \frac{2a}{\lambda^2 - 4} \{\lambda \cos(\lambda\alpha + M) \cos 2\alpha + 2 \sin(\lambda\alpha + M) \sin 2\alpha\}$$

$$- L \frac{2a}{\lambda} \cos(\lambda\alpha + M) + H.$$

If we suppose the two supports to be on the same horizontal line, we must have $\xi = 0$ and $\eta = 0$, when $\alpha = \pm\alpha_0$. These conditions may be satisfied if we take $M = \frac{\pi}{2}$, $H = 0$, for then ξ becomes an even and η an odd function of α. In this case

$\eta = 0$ at the lowest point of the chain. We have then two equations to find $\frac{L}{h}$, equating these values, we have

$$\frac{2\tan 2\alpha_0 - \lambda \tan\lambda\alpha_0 + \frac{\tan\lambda\alpha_0}{\cos 2\alpha}\,\frac{\lambda^2-4}{\lambda}}{2\alpha_0 + \sin 2\alpha_0} = \frac{\lambda \tan\lambda\alpha_0 \tan 2\alpha_0 + 2}{2\cos^2\alpha_0 + \frac{4}{\lambda^2-4}}.$$

If α_0 be small, this equation is very nearly satisfied by $\lambda\alpha_0 = i\pi$ where i is any integer.

If the two supports are not in the same horizontal line, the formulæ are much more complicated.

15. The equation (23) to determinine ϕ' can be integrated for another law of density of the chain. The coefficient of ϕ' vanishes if $\mu' = B\sin(2\alpha + c)$ where B and c are any constants. In this case the intrinsic equation to the curve in which the string hangs in equilibrium is

$$\frac{\cos\alpha}{\rho} = \frac{B^4}{g}\sin^4(2\alpha + c),$$

and by (12) the law of density is given by

$$m = \frac{AB^4}{g}\cdot\frac{\sin^4(2\alpha + c)}{\cos^3\alpha}.$$

Supposing the motion such that $f(t) = 0$, we have

$$\phi' = F\left(t + \int\frac{d\alpha}{\mu}\right) + \psi\left(t - \int\frac{d\alpha}{\mu}\right).$$

Each of these terms represents a separate wave, taking only the first term, we have

$$\phi = \sqrt{(\mu)}\,F\left(t + \int\frac{d\alpha}{\mu}\right).$$

Since $\mu = B^2\sin^2(2\alpha + c)$ this may be put under the form

$$\phi = B\sin(2\alpha + c)\,F\left\{t - \frac{\cot(2\alpha + c)}{2B^2}\right\}.$$

To find X the displacement of any particle of the string along the tangent, we have, by (19),

$$\frac{d^2X}{d\alpha^2} + X = \rho\phi$$

$$= \frac{g}{B^3}\,\frac{\cos\alpha}{\sin^3(2\alpha + c)}\,F\left\{t - \frac{\cot(2\alpha + c)}{2B^2}\right\}.$$

This can be easily integrated in the case in which $c = -\pi$. Multiplying by $\sin\alpha$ and integrating, we get

$$\sin\alpha\,\frac{dX}{d\alpha} - \cos\alpha X = \frac{g}{2B}F_1\left\{t - \frac{\cot 2\alpha}{2B^2}\right\} - A,$$

where A is a function of t. But A is not really arbitrary, for we have put $f(t) = 0$, and the equation (20) is not satisfied unless $\dfrac{d^2A}{dt^2} = 0$. Including the A in the functional integral, we have

$$\frac{X}{\sin\alpha} = \frac{g}{2B}\int \frac{F_1\left(t - \dfrac{\cot 2\alpha}{2B^2}\right)}{\sin^2\alpha}\,d\alpha.$$

Also, if ξ be, as before, the horizontal displacement of any particle of the string, we have, by (17),

$$\begin{aligned}\xi &= -\sin\alpha\,\frac{dX}{d\alpha} + X\cos\alpha \\ &= -\frac{g}{2B}F_1\left(t - \frac{\cot 2\alpha}{2B^2}\right).\end{aligned}$$

Thus the complete values of ϕ and ξ are

$$\phi = -B\sin 2\alpha F\left(t - \frac{\cot 2\alpha}{2B^2}\right) - B'\sin 2\alpha\psi\left(t + \frac{\cot 2\alpha}{2B^2}\right),$$

$$\xi = \frac{g}{2B}F_1\left(t - \frac{\cot 2\alpha}{2B^2}\right) - \frac{g}{2B'}\psi_1\left(t + \frac{\cot 2\alpha}{2B^2}\right).$$

These formulæ apply to any cases in which the motion is symmetrical about a vertical line.

If the two points of support A, B of the string be very nearly in the same vertical line, so that the string is very nearly vertical, the inclination α is very nearly equal to $\dfrac{\pi}{2}$. Let us put $\alpha = \dfrac{\pi}{2} - \beta$. Then neglecting the higher powers of β, the equations of equilibrium become after integration

$$\frac{1}{\beta^2} = E(s + l'),$$

where E and l' are two constants. Hence the density of the string is

$$m = \frac{A'}{\sqrt{(s + l')}},$$

where A' is another constant. This is the same law of density which we have already discussed in Art. 6 for the case of a chain suspended solely by one end.

EXERCISES IN THE INTEGRAL CALCULUS.

(Continued from p. 12).

By *Sir J. Cockle, F.R.S.*

§ 2. *On certain Leading Equations.*

13. By analogy with the notation

$$\frac{dz}{dx} = p, \quad \frac{d^2z}{dx^2} = \frac{dp}{dx} = r,$$

I shall put

$$\frac{dz}{d\xi} = \pi, \quad \frac{d^2z}{d\xi^2} = \frac{d\pi}{d\xi} = \rho,$$

and, this being done, from the equation

$$2\xi\frac{d^2z}{d\xi^2} + (2a+1)\frac{dz}{d\xi} + bz = 0 \;\ldots\ldots\ldots\ldots (15),$$

we deduce, by successive differentiations,

$$2\xi\frac{d^2\pi}{d\xi^2} + \{2(a+1)+1\}\frac{d\pi}{d\xi} + b\pi = 0 \;\ldots\ldots (16),$$

$$2\xi\frac{d^2\rho}{d\xi^2} + \{2(a+2)+1\}\frac{d\rho}{d\xi} + b\rho = 0 \;\ldots\ldots (17),$$

$$\&c. = \&c. = \&c.$$

Let $z = \phi(a)$ represent the solution of (15). Then $\pi = \phi(a+1)$ will be the solution of (16) and $\rho = \phi(a+2)$ will be the solution of (17), and so onwards. Hence, in general, n being an integer,

$$\frac{d^n\phi(a)}{d\xi^n} = \phi(a+n) \;\ldots\ldots\ldots\ldots\ldots\ldots (18).$$

14. Next, let

$$\beta = (2\xi)^{a+\frac{1}{2}}\pi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

then
$$2\xi\frac{d^2\beta}{d\xi^2} = (2\xi)^{a+\frac{3}{2}}\frac{d^2\pi}{d\xi^2} + 2(2a+1)(2\xi)^{a+\frac{1}{2}}\frac{d\pi}{d\xi}$$
$$+ (2a+1)(2a-1)(2\xi)^{a-\frac{1}{2}}\pi \;\ldots\ldots\ldots (20),$$

$$(1-2a)\frac{d\beta}{d\xi} = (1-2a)(2\xi)^{a+\frac{1}{2}}\frac{d\pi}{d\xi}$$
$$+ (1-2a)(2a+1)(2\xi)^{a-\frac{1}{2}}\pi \;\ldots\ldots\ldots (21),$$

$$b\beta = b(2\xi)^{a+\frac{1}{2}}\pi \;\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

Hence, adding (20), (21), and (22), we have

$$2\xi \frac{d^2\beta}{d\xi^2} + (1-2a)\frac{d\beta}{d\xi} + b\beta$$

$$= (2\xi)^{a+\frac{1}{2}} \left\{2\xi \frac{d^2\pi}{d\xi^2} + [2(a+1)+1]\frac{d\pi}{d\xi} + b\pi\right\} \ldots\ldots (23).$$

Now the last number of (23) vanishes, in virtue of (16). And, according to our notation, the solution of (23) is

$$\beta = \phi(-a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (24).$$

Consequently, by (19),

$$(2\xi)^{a+\frac{1}{2}}\pi = (2\xi)^{a+\frac{1}{2}}\phi(a+1) = \phi(-a),$$

a result which may be put under the symmetrical form

$$(2\xi)^{\frac{1}{2}(a+1)}\phi(a+1) = (2\xi)^{-\frac{1}{2}a}\phi(-a) \ldots\ldots\ldots (25),$$

or

$$\psi(a+1) = \psi(-a) \ldots\ldots\ldots\ldots\ldots\ldots (26).$$

Hence $\phi(a)$ possesses the independent properties indicated by (18) and (25).

15. In general (15) can be solved by a definite integral or integrals. There are cases in which it admits of finite solution. In (18) let $a=0$. Then we have

$$\frac{d^n\phi(0)}{d\xi^n} = \phi(n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (27),$$

and, n being positive, we may deduce $\phi(n)$ from $\phi(0)$ by differentiation. If n be negative (25) gives us

$$\phi(-n) = (2\xi)^{\frac{1}{2}(2n+1)}\phi(n+1) \ldots\ldots\ldots\ldots (28),$$

and enables us to avoid integrations, though (27) holds for negative values of n. If we use (27) for such negative values care must be taken with respect to the introduction of arbitrary constants. Now $\phi(0)$ is the solution of the equation obtained by putting $a=0$ in (15); viz.

$$2\xi \frac{d^2z}{d\xi^2} + \frac{dz}{d\xi} + bz = 0 \ldots\ldots\ldots\ldots\ldots (29).$$

Dividing (29) by 2ξ, it becomes

$$\frac{d^2z}{d\xi^2} + \frac{1}{2\xi}\frac{dz}{d\xi} + \frac{b}{2\xi}z = 0,$$

which is soluble by a change of the independent variable, or, more readily, by reverting to (29), dividing it by 2 and then writing its symbolical decomposition (wherein $b_2 = -b$)

$$\left\{\xi^{\frac{1}{2}} \frac{d}{d\xi} \mp \sqrt{\left(\frac{b_2}{2}\right)}\right\} \left\{\xi^{\frac{1}{2}} \frac{d}{d\xi} \pm \sqrt{\left(\frac{b_2}{2}\right)}\right\} z = 0.$$

This decomposition shews that z is obtainable from

$$\frac{dz}{d\xi} \pm z \sqrt{\left(\frac{b_2}{2\xi}\right)} = 0,$$

whence we find

$$z = C_1 e^{\sqrt{(2b_2\xi)}} + C_{-1} e^{-\sqrt{(2b_2\xi)}} \quad \text{.............. (30).}$$

16. Properties corresponding to (18) and (25) reappear in transformations of (15). Let

$$\xi = \frac{x^2}{2}, \text{ whence } \frac{d\xi}{dx} = x, \ \frac{d^2\xi}{dx^2} = 1,$$

then, writing (15) in the form

$$\frac{d^2z}{d\xi^2} + \left(\frac{2a+1}{2\xi}\right) \frac{dz}{d\xi} + \frac{bz}{2\xi} = 0,$$

we see that a change of the independent variable from ξ to x changes (15) into

$$\frac{d^2z}{dx^2} + \frac{2a}{x}\frac{dz}{dx} + bz = 0 \quad \text{............... (31),}$$

an equation of the same class as that the discussion of which extends from pages 463 to 475 of Boole's work, and elsewhere* therein. And since

$$d\xi = xdx,$$

we now see that, if $\phi(a)$ denote a solution of (31), then, in virtue of (18) and (25) respectively,

$$\phi(a+n) = \left(\frac{1}{x}\frac{d}{dx}\right)^n \phi(a) \quad \text{............ (32),}$$

$$x^{a+1}\phi(a+1) = x^{-a}\phi(-a) \quad \text{............... (33),}$$

wherein $\left(\frac{1}{x}\frac{d}{dx}\right)^n = \left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{1}{x}\frac{d}{dx}\right) \ldots (n \text{ factors}) = \left(\frac{d}{d\xi}\right)^n,$

* See Ex. 14 of p. 438, Ex. 15 of p. 439, Ex. 1 of p. 457, Ex. 4 of p. 458 of Boole. By criticoids, Ex. 16 of p. 440 is reducible to the same class.

which differs materially from $\frac{1}{x^n}\left(\frac{d}{dx}\right)^n$. Compare Boole, p. 398.

17. That the roots of (31) possess the properties (32) and (33) is readily shewn by criticoids, which indeed mainly or entirely led me to them. And it was through the properties of (31) that I was conducted to those of (15). The latter properties are virtually contained in Arts. 93, 94, and 95 of my "Notes on the Differential Calculus," *supra*, vol. III., pp. 252–253. Art. 96 of the same "Notes," (*ibid.*, p. 253), indicates the soluble cases. But we have, either directly from (31), or by substitution in (30)

$$\phi(0) = C_1 e^{x\sqrt{(-b)}} + C_{-1} e^{-x\sqrt{(-b)}} \text{ .... } (34).$$

"Oakwal," near Brisbane, Queensland,
Australia, *December* 21*st*, 1870.

AN ANGLE-PROPERTY OF THE RIGHT CIRCULAR CONE.

By *C. Taylor, M.A.*

WHEN we commence by regarding curves of the second order as sections of the right circular cone we arrive very simply at some of their most interesting line-properties, but we are not accustomed to deduce their angle-properties directly from the cone. Now, without considering at all the general question whether or not it is desirable to bring in properties of solids in order to deduce therefrom properties of plane figures, I propose to shew how the fundamental angle-properties of conic sections may be deduced directly from the cone, viz. with the help of a simple property of that solid which does not appear explicitly in the ordinary text books, but is arrived at by an easy development from what is given therein. The property in question is as follows:

(1) *At any point on a plane section of a right circular cone the tangent is equally inclined to the focal distance and the generating line.*

It is known that if spheres be inscribed in a cone so as to touch any secant plane, their points of contact with that

plane are the foci of the curve in which it cuts the cone. Let spheres, so described, touch the plane of section in S, S', (fig. 6) and let the generating line drawn from the vertex V to any point P of the section meet the plane of contact of the S-sphere in p. Then Vp is constant throughout the circle of contact. Also $Pp = PS$, since they are tangents to the sphere.

Therefore $$SP + \text{a constant} = VP.$$

Therefore, taking an adjacent point P' (fig. 7) on the section,

$$SP \sim SP' = VP \sim VP',$$

or the increments of SP, VP are equal.

On VP', SP' drop perpendiculars Pm, Pn. Then ultimately $P'm$, $P'n$ are the increments of SP, VP. Therefore $P'm = P'n$; also PP' is common to the triangles $P'mP$, $P'nP$, and the angles at m, n are right angles.

Therefore $\angle VP'P = SP'P$ ultimately; or if PR (fig. 6) be the tangent at P, then

$$\angle SPR = VPR.$$

(2) It follows that in the right circular cylinder

$$\angle SPR = \infty PR$$

$$= \text{the inclination of } PR \text{ to the axis of the cylinder.}$$

We may notice the following verification:—Take PR parallel to the axis of the section; then P coincides with B, and $\angle SPR = \sin^{-1} \dfrac{b}{a}$, with the usual notation. And this, as we know, is also the angle between the axes of the cylinder and the section.

(3) We may also prove the foregoing theorem without using the method of limits. For if from any point R (fig. 6) in the tangent at P, Rp be drawn, viz. to the point in which the generating line through P meets the plane of contact of the S-sphere, then Rp is a tangent to the sphere, and therefore $RS = Rp$. Similarly $PS = Pp$. Hence the triangles PSR, PpR, having the third side PR common, are equal in all respects.

Therefore $$\angle SPR = PpR.$$

$$\angle SRP = pRP.$$

$$\angle PSR = PpR.$$

Moreover these relations will be seen to hold when P, like R, is *any point whatever* on the tangent.

Hence we may deduce some of the plane angle-properties of conics.

(A) *At any point the tangent is equally inclined to the focal distances.*

For since (fig. 6)

$$\angle SPR = VPR,$$

and similarly, for the focus S', supposing VP, RP produced to V', R',

$$\angle S'PR' = V'PR';$$

therefore, the vertical angles at P being equal,

$$\angle SPR = S'PR'.$$

(B) *The tangent measured from the point of contact to the directrix subtends a right angle at the focus.*

For convenience of expression suppose the axis of the cone vertical. Let PR (fig. 6) meet the horizontal tangent through p in R; then R lies on the common section of the planes of the conic and the circle; that is to say, R *lies on the directrix.*

Also, as above, $\angle PSR = PpR$

$$= \textit{a right angle.}$$

(C) *Any two tangents measured from their point of intersection to their points of contact subtend equal (or supplementary) angles at the focus.*

Through any pair of tangents PT, $P'T$ to the conic (fig. 8) drawn tangent planes meeting the plane of contact in the horizontal tangents pt, $p't$. Then $tp = tp'$, and similarly $Tp = Tp'$. Also tT is common to the two triangles ptT, $p'tT$. Therefore the angles tpT, $tp'T$ are equal. Therefore their complements TpP, $Tp'P'$ are equal, and these (see § 3) are equal to the angles subtended by TP, TP' at S.

Thus much will have sufficed to indicate a method of treating the angle-properties of conics regarded as sections of the cone. Of the examples which have been given, I remark in conclusion that (A) has its advantages from a particular point of view, for it gives a *direct* proof of the property in question, without requiring us to assume the limit-definition of a tangent.

June, 1871.

ON AN ANALOGUE IN THE THEORY OF QUADRICS TO A KNOWN PROPERTY IN THE THEORY OF CONICS.

By *R. Townsend, M.A., F.R.S.*

It is a well-known property, in the theory of conics, that

A variable chord of a fixed conic, cut in a constant anharmonic ratio by two fixed tangents to the curve, envelopes one of two other conics having double contact with the original at its points of contact with the tangents, and coinciding with each other in the case of harmonic section.

So, analogously, in the theory of quadrics,

A variable chord of a fixed quadric, cut in a constant anharmonic ratio by two fixed tangents to the surface, generates one of two other quadrics having double contact with the original at its points of contact with the tangents, and coinciding with each other in the case of harmonic section.

Which may be easily shewn as follows: Taking, as the four planes of reference, the two tangent planes A and B to the surface at the two points of contact P and Q of the two tangents PX and QY, and the two planes C and D determined by the two tangents PX and QY with their chord of contact PQ; the equation of the quadric being then of the form $m.\alpha\beta + n.\gamma\delta + p.\gamma^2 + q.\delta^2 = 0$, if bd and ac be the coordinates of any two points X and Y on the two tangents PX and QY, we have, to determine the ratio $\mu : \nu$ of the section of their line of connection XY by the surface, the equation $pc^2.\mu^2 + (mab + ncd).\mu\nu + qd^2.\nu^2 = 0$, and therefore, as the condition of the constancy of the anharmonic ratio of the two points of section, the relation $m.ab + n.cd = \pm k\sqrt{(pq)}.cd$, where $k = 0$ in the case of harmonic section; from which, if $\alpha, \beta, \gamma, \delta$ be the coordinates of any point on the line XY, since evidently $\alpha : \gamma = a : c$ and $\beta : \delta = b : d$, it follows at once that $m.\alpha\beta + n.\gamma\delta = \pm k\sqrt{(pq)}.\gamma\delta$, which, representing two quadrics having double contact with the original at the two points P and Q, therefore, &c.

Trinity College, Dublin,
July 8, 1871.

SOLUTIONS OF A SMITH'S PRIZE PAPER FOR 1871.

(Continued from p. 47).

By Professor *Cayley*.

7. *The coordinates* (x, y, z, w) *of a point* P *in space are connected with the coordinates* (x', y', z') *of a point* P' *in a plane by the equations*

$$x : y : z : w = X' : Y' : Z' : W',$$

where X', Y', Z', W' *are quadric functions of* (x', y', z') *such that* $X' = 0$, $Y' = 0$, $Z' = 0$, $W' = 0$ *represent conics having a common point: shew that the locus of* P *is a cubic scroll (skew surface of the third order): and find the curves in the plane which correspond to the generating lines of the scroll.*

The equations $x : y : z : w = X' : Y' : Z' : W'$ are three equations containing the indeterminate parameters $x' : z'$ and $y' : z'$, so that eliminating these we have between (x, y, z, w) a single (homogeneous) equation representing a surface. To each point (x', y', z') of the plane, there corresponds a single point of the surface, and to each point (x, y, z, w) of the surface a single point (x', y', z') of the plane. The only exception is that for the common point of the four conics, the ratios $x : y : z : w$ are essentially indeterminate, and there is not corresponding hereto any determinate point of the surface.

To find the order of the surface, consider its intersection with any arbitrary line

$$ax + by + cz + dw = 0,$$

$$a_1x + b_1y + c_1z + d_1w = 0.$$

We have corresponding hereto in the plane the points of intersection of the conics

$$aX' + bY' + cZ' + dW' = 0,$$

$$a_1X' + b_1Y' + c_1Z' + d_1W' = 0,$$

viz. these are conics each of them passing through the common point of the four conics, and therefore intersecting besides in three points: that is the order of the surface is $= 3$.

[To shew that the common point to be (as above) excluded, some further explanation is desirable. To the section of the surface by the plane $ax+by+cz+dw=0$, corresponds the conic $aX'+bY'+cZ'+dW'=0$; and similarly to the section by the plane $a_1x+b_1y+c_1z+d_1w=0$, corresponds the conic $a_1X'+b_1Y'+c_1Z'+d_1W'=0$. Now to the common point *considered as belonging to the first conic* there corresponds a determinate point of the surface; and to the common point *considered as belonging to the second conic* there corresponds a determinate point of the surface; but these are two distinct points on the surface: so that corresponding to the common point of the four conics, there is not on the surface any point of intersection of the two plane sections; but these intersect in only three points of the surface; viz. the line of intersection of the two planes meets the surface in three points: or the surface is a cubic surface.]

The same result may be obtained, and it may be further shewn that the surface is a scroll, by means of the property in the foregoing question 6; viz. it thereby appears that each of the functions X', Y', Z', W' may be taken to be of the form $ax'^2+by'^2+fy'z'+gz'x'$; hence replacing the original coordinates x, y, z, w, by properly selected linear functions of these coordinates, the given relations may be presented in the form

$$x:y:z:w=x'^2:y'^2:x'z':y'z',$$

whence eliminating, we have

$$xw^2-yz^2=0$$

the equation of a cubic scroll, having the line $z=0$, $w=0$ for a double line, and the line $x=0$, $y=0$ for a directrix line. The equations of a generating line of the scroll are, it is clear, $z-\theta w=0$, $x-\theta^2y=0$, where θ is a variable parameter; and corresponding hereto in the plane we have the line $x'-\theta y'=0$, viz. this is any line through the common intersection of the four conics.

8. *If U, V are binary functions of the form $(a, b, \ldots)(x, y)^m$ with arbitrary coefficients, and if the equations $U=0$, $V=0$ have a common root, shew how this can be determined in terms of the derived functions of the Resultant in regard to the coefficients of either function.*

Shew what results in regard to the common root can be obtained when the coefficients are not all of them arbitrary

but (1) *each or either of the functions depends in any manner whatever on a set of arbitrary coefficients not entering into the other function*, (2) *the two functions depend in any manner whatever on one and the same set of arbitrary coefficients.*

How is the theory modified when, instead of the two equations, there is a single equation $U=0$ *having a double root.*

Suppose

$$U=(a,\ b,\ \ldots)(x,\ y)^m\left(=ax^m+\frac{m}{1}\,bx^{m-1}y+\&\text{c.}\right),$$

$$V=(a',\ b',\ \ldots)(x,\ y)^{m'}\left(=a'x^{m'}+\frac{m'}{1}\,bx^{m'-1}y'+\&\text{c.}\right).$$

Then if R is the resultant, the equation $R=0$ is the relation which must exist between the coefficients $(a,\ b,\ \ldots)$ and $(a',\ b',\ \ldots)$ in order that the equations $U=0$ and $V=0$ may have a common root (that is, in order that the functions U, V may have a common factor $x-\alpha y$). Imagine the relation subsisting, and that x, y are the values belonging to the common root, or (what is the same thing) that we have $x-\alpha y=0$; we have then simultaneously $U=0$, $V=0$, $R=0$. Now suppose the coefficients a, b,... to be infinitesimally varied in such manner that U, V have still a common root; say the new values are $a+\delta a$, $b+\delta b$, ...: this implies between δa, δb, ... the relation

$$\frac{dR}{da}\,\delta a+\frac{dR}{db}\,\delta b+\ldots=0.$$

But the common factor $x-\alpha y$ is a factor of the *unaltered* equation $V=0$, and the values of $(x,\ y)$ are thus unaltered, viz. the equation $U=0$ is satisfied with the original values of $(x,\ y)$; and we thus have

$$\frac{dU}{da}\,\delta a+\frac{dU}{db}\,\delta b+\ldots=0,$$

or what is the same thing

$$x^m\delta a+mx^{m-1}y\delta b+\ldots=0,$$

an equation which must agree with the former one, that is we have

$$x^m : mx^{m-1}y : \&\text{c.}=\frac{dR}{da}:\frac{dR}{db}:\&\text{c.},$$

a series of equations giving the value of the common root $\frac{x}{y}$ ($=\alpha$) in the several forms

$$\frac{1}{m}\frac{x}{y}=\frac{dR}{da}\div\frac{dR}{db},\quad \frac{2}{m-1}\frac{x}{y}=\frac{dR}{db}\div\frac{dR}{dc},\ \&c.$$

and it is clear that we have in like manner

$$x^{m'} : m'x^{m'-1}y : \&c. = \frac{dR}{da'} : \frac{dR}{db'} : \&c.$$

It is clear that if U involves, *in any manner whatever*, the coefficients $a, b,\ldots$ which do not enter into the function V, then we have in precisely the same manner

$$\frac{dU}{da} : \frac{dU}{db} : \&c. = \frac{dR}{da} : \frac{dR}{db} : \&c.,$$

a system of equations satisfied by the values x, y which belong to the common root.

But if the coefficients $a, b,\ldots$ are contained in any manner whatever in both of the functions U, V; then by altering $a, b,\ldots$ we alter the common root; say that $x+\delta x$, $y+\delta y$ belong to its new value; then we have

$$\frac{dU}{dx}\delta x+\frac{dU}{dy}\delta y+\frac{dU}{da}\delta a+\frac{dU}{db}\delta b+\ldots=0,$$

$$\frac{dV}{dx}\delta x+\frac{dV}{dy}\delta y+\frac{dV}{da}\delta a+\frac{dV}{db}\delta b+\ldots=0.$$

Now the values of x, y which satisfy $U=0$, $V=0$ also satisfy

$$\frac{dU}{dx}\frac{dV}{dy}-\frac{dU}{dy}\frac{dV}{dx}=0,$$

hence from the foregoing two equations eliminating δx or δy, the other of these two quantities will disappear of itself, and we thus obtain an equation

$$A\delta a+B\delta b+\ldots=0,$$

which must agree with the above equation

$$\frac{dR}{da}\delta a+\frac{dR}{db}\delta b+\ldots=0,$$

or we have

$$\frac{dR}{da} : \frac{dR}{db} : \&c. = A : B : \&c.,$$

a system of equations satisfied by the values x, y which belong to the common root.

In the case of a single equation $U=0$ having a double root, the condition for this is $\Delta=0$, where Δ is the discriminant of the function U; and the like reasoning shows that for the values x, y which belong to the double root we have

$$\frac{dU}{da} : \frac{dU}{db} : \&c. = \frac{d\Delta}{da} : \frac{d\Delta}{db} : \ldots ;$$

viz. if U is of the form $(a, b, \ldots)(x, y)^m$ with arbitrary coefficients, then we have thus a series of equations giving the required value of $\frac{x}{y}$; but if $(a, b, \ldots)$ are arbitrary coefficients contained in any manner whatever in the function U, then we have a series of equations satisfied by the values x, y which belong to the double root.

9. *The normal at each point of a principal section of an ellipsoid is intersected by the normal at a consecutive point not on the principal section: shew that the locus of the point of intersection is an ellipse having four (real or imaginary) contacts with the evolute of the principal section.*

The principal section is for convenience taken to be that in the plane of zx; the coordinates of any point thereof are therefore X, 0, Z where

$$\frac{X^2}{a^2} + \frac{Y^2}{c^2} = 1.$$

Consider the normal at a point X, Y, Z of the ellipsoid; taking x, y, z as current coordinates, the equations of the normal are

$$\frac{x-X}{\frac{X}{a^2}} = \frac{y-Q}{\frac{Y}{b^2}} = \frac{z-Z}{\frac{Z}{c^2}}.$$

Writing herein $y=0$, we have

$$x = X\left(1-\frac{b^2}{a^2}\right), \quad z = Z\left(1-\frac{b^2}{c^2}\right);$$

viz. x, z are here the coordinates of the point where the normal meets the plane of xz; and observing that the point in question lies on the normal at the point X, 0, Z, it is

clear that x, y, z will be the coordinates of the intersection of the last-mentioned normal by the normal at the consecutive point not on the principal section.

Writing for shortness

$$\alpha = b^2 - c^2,\quad \beta = c^2 - a^2,\quad \gamma = a^2 - b^2,$$

($\alpha + \beta + \gamma = 0$, α and γ positive, β negative) the values are

$$x = \frac{\gamma X}{a^2},\quad z = -\frac{\alpha Z}{c^2},$$

wherefore
$$\frac{X}{a} = \frac{ax}{\gamma},\quad \frac{Z}{c} = -\frac{cz}{\alpha};$$

or substituting in
$$\frac{X^2}{a^2} + \frac{Z^2}{c^2} = 1,$$

we have
$$\frac{a^2x^2}{\gamma^2} + \frac{c^2z^2}{\alpha^2} = 1,$$

the required locus, which is thus an ellipse.

If the point $(X, 0, Z)$ is an umbilicus it is clear that the corresponding point of the locus will be a point of the evolute of the principal section; and to prove that the locus touches the evolute, it is only necessary to shew that the tangent of the locus is also the tangent of the evolute; or what is the same thing, that the tangent of the locus passes through the umbilicus.

Now for the umbilicus we have

$$X^2 = -a^2\frac{\gamma}{\beta},\quad Z^2 = -c^2\frac{\alpha}{\beta};$$

the corresponding values of x, z being

$$x = \frac{\gamma X}{a^2},\quad z = -\frac{\alpha Z}{c^2}.$$

Take ξ, ζ as the current coordinates of a point on the tangent of the locus, we have

$$\frac{a^2x\xi}{\gamma^2} + \frac{c^2x\zeta}{\alpha^2} = 1,$$

or substituting for x, z the foregoing values,

$$\frac{X\xi}{\gamma} - \frac{Z\zeta}{\alpha} = 1,$$

and these should be satisfied by $\xi, \zeta = X, Z$; viz. we ought to have

$$\frac{X^2}{\gamma} - \frac{Z^2}{\alpha} = 1,$$

and this equation is in fact true for the values of X, Z at the umbilicus; viz. for these values we have

$$-\frac{a^2}{\beta} + \frac{c^2}{\beta} = 1,$$

that is $\beta = c^2 - a^2$, which is in fact the value of β.

There is obviously a contact in each quadrant, that is there are four contacts (in the present case all real) of the locus with the evolute.

The same theorem holds good in regard to the other principal sections; only for these, the umbilici being imaginary, the points of contact of the locus with the evolute are also imaginary.

Remark. There is a great convenience in questions relating to the ellipsoid, in the use of the foregoing notations α, β, γ.

(To be continued).

REVIEWS.

Jahrbuch über die Fortschritte der Mathematik im Verein mit andern Mathematikern herausgegeben. Von Dr. CARL OHRTMANN und Dr. FELIX MÜLLER. Erster Band. Jahrgang, 1868 (in 3 Heften), Heft. 2. (Berlin: George Reimer, 1871.)

This forms a portion of a work similar to the well-known Fortschritte der Physik, and gives the title, accompanied by a very brief account, of every mathematical memoir, paper, or book that has appeared in the year anywhere over the globe. The arrangement is according to subjects, and the present part contains the eighth and ninth divisions, viz. Analytical and Synthetical Geometry. The use and the value of the work are undoubted, but the notices of the papers are often so short as to be of very slight use. This year (1868) will form the first volume of the Fortschritte; after its completion, we believe, 1869 and 1870 will be speedily published and then the work will appear annually as soon after the completion of each year as possible.

A new table of seven-place logarithms of all numbers from 20,000 *to* 200,000. By EDWARD SANG, F.R.S.E. (London: Charles and Edwin Layton, 1871.)

The present work marks an epoch in the history of logarithms, as it is nearly 250 years since the publication of the last extensive table of the logarithms of numbers from original computations. The history of the calculations, by which were obtained the common logarithms, since reprinted so frequently and in so many different forms, may be told in very few words.

In 1614 Napier published his Mirifici Logarithmorum Canonis Descriptio, containing a table of logarithms accompanied by an explanation (the logarithms

were not quite the same as those now called Napierian). This work Briggs, at that time Professor at Gresham College, and afterwards Savilian Professor at Oxford, so much admired that he resolved to visit Napier: "Napier lord of Markinston hath set my head and hands at work with his new and admirable logarithms: I hope to see him this summer, if it please God; for I never saw a book which pleased me better, and made me more wonder"; this he says in a letter to Archbishop Usher. Briggs accordingly visited Napier at Edinburgh and stayed with him a whole month, having previously written to him and suggested the advantage of 10 as the base of a system of logarithms. There is a slight doubt as to who really did first notice the advantage of this base, but it seems most probable that Briggs was really the inventor of common logarithms and that Napier scarcely acted quite fairly in the matter. Whether or no Briggs was really the originator of the system, there is no doubt that to him is due the calculation of the first table of logarithms to base 10, which he printed, after his return from a second visit to Napier in 1617, under the title Logarithmorum Chilias Prima. This work, which contains the common logarithms of the first thousand numbers to eight places of decimals, as we should now say, was not published till 1618, after the death of Napier. Briggs continued to labour unceasingly at the calculation of logarithms, and in 1624 published his Arithmetica Logarithmica, which contained the logarithms of 30,000 numbers, viz. the numbers from 1 to 20,000 and from 90,000 to 100,000, to 14* places of decimals. In his preface he earnestly solicits others to fill up the gap and offers to give instructions and ruled paper to any one willing to help to complete the tables.

In 1628 Adrian Vlacq published at Gouda his Arithmetica Logarithmica, containing the logarithms of all numbers from 1 to 100,000 to 10 places of decimals, Briggs having been employed in the interval in computing logarithmic sines and cosines.

Vlacq styled his work a second edition of Briggs' Arithmetica Logarithmica, a title that gave great offence in England. Thus Norwood writes in his *Trigonometrie* that when he mentions the Arithmetica Logarithmica "you are to understand not the book put forth about a month since in English, as a translation of his, and with the same title; being nothing like his, nor worthy of his name......And here I have just occasion to blame the ill dealing of these men both in the matter before mentioned and in printing a second edition of his Arithmetica Logarithmica in Latin whilst he lived, against his mind and liking......" The English book referred to consisted of Vlacq's tables with an English introduction, published by George Miller, a work now far more common than Vlacq's Latin edition.

Vlacq was undoubtedly wrong in calling his work a second edition of Briggs', and publishing it without consulting the latter, but his fault as De Morgan remarked was one of 'over-ascription,' not of plagiarism. He calculated 70,000 logarithms and copied 30,000, so that he would have been justified in calling the work his own. It is however, to be regretted that he did not consult Briggs before publication. From this time (1628) till the end of the last century no common logarithms of numbers were calculated, although edition after edition of tables of logarithms of all sizes and modes of arrangement was being continually published. Vlacq's work may in fact be said to have been the father of every table that has been published subsequently till the appearance of the work now under notice. De Morgan, who only professes to notice works of some importance, has enumerated more than 50 tables, and several of these went through many editions.

In 1794, at the suggestion of Carnot and others, the French revolutionary government ordered the calculation of very extensive tables; Prony, who was associated with several other mathematicians, including Legendre, was charged with the superintendence of the calculations, and he was expressly directed "non-seulement à composer des Tables qui ne laissassent rien à désirer quant à l'exactitude, *mais à en faire le monument de calcul le plus vaste et le plus impossant qui eût jamais été exécuté ou même conçu.*" This programme was carried

---

* It has often been stated that Briggs's table gives the logarithms to 15 and Vlacq's to 11 decimal places; the errors are caused by the characteristic—now universally omitted—having been counted as a decimal place.

out; and the results, upon the calculation of which from 60 to 80 computers were engaged for two years, fill seventeen very large folios in manuscript. The whole work was performed in duplicate, most of the results being obtained by interpolation by finite-difference formulæ. Of the seventeen volumes, eight contain the logarithms of numbers from 1 to 200,000 to 15 places of decimals and the rest are devoted to natural and logarithmic sines, cosines, tangents &c., divided centesimally. Although the logarithms extend to 15 places, M. Lefort, who has recently examined the tables and published a most interesting account of his investigations in the Annales de l'Observatoire de Paris, states that 12 places only (which is as many places as the tables were intended to extend to) can be relied on, and that the 13th place may be in error by 8. The tables, as is well known, have never been published but were deposited in the library of the Observatory where they still remain.* Mr. Babbage, when preparing his well-known seven figure table, visited Paris and consulted them, and M. Lefort has, in the memoir cited above, given a table of errata in Briggs' and Vlacq's tables, which he had found by comparison of their works with the Tables du Cadastre. With these exceptions, no use, as far as we know, has been made of these elaborate tables, and their non-publication is much to be regretted. "In 1820," says De Morgan, "a distinguished member of the Board of Longitude, London, was instructed by our government to propose to the Board of Longitude, of Paris, to print an abridgment of these tables at the joint expense of the two countries: £5000 was named as the sum which our Government was willing to advance for this purpose, but the proposal was declined."†

For the preparation of Mr. Sang's Table, "it was requisite to calculate the logarithms of the hundred thousand numbers from 100,000 to 200,000. For this purpose the whole calculation has been made from the beginning as if logarithms had never before been computed; and each step of the calculation has been recorded for use in case of suspected error......the manuscript exhibits, to 15 places, the logarithms of all these numbers [from 100,000 to 200,000] with their first and second differences, and with scarce the probability of a single error throughout, except, indeed, the unavoidable minute errors in the fifteenth place. Thus the amount of work done in preparing the new part of this table has been greater than the labours of Briggs and Vlacq conjoined, even although the facilities afforded by modern methods be taken into account."

Mr. Sang has therefore calculated *de novo* the logarithms to 15 places of numbers from 100,000 to 200,000, and has thus performed probably the largest piece of numerical work ever undertaken by a single individual, and every mathematician and 'friend of exact science,' must congratulate him most sincerely on its accomplishment. The publication or non-publication of eight of the seventeen volumes of the Tables du Cadastre is a matter of far less interest now than before, and Mr. Sang's manuscript is more valuable than the corresponding portion of the French tables. The object Mr. Sang had in view in tabulating the logarithms of numbers from 20,000 to 200,000 instead of from 10,000 to 100,000 is thus explained in the preface. 'At the beginning of the seven-place table [in ordinary tables of logarithms] the differences are large and numerous, the side tables of proportional parts are crammed into the page, and hence the interpolations can be performed mentally only by experienced computers. As we advance in the table, the differences become smaller and fewer, so that the interpolation is easily managed: the object of the present table is to secure this facility all along." The small numbers commencing 10...... which usually come at the commencement of the table here come in the middle, and 'the differences are halved in magnitude, while the number of them in a page is quartered.' The great advantage Mr. Sang's table possesses over the usual ones, Babbage, Callet &c., is rendered evident at a glance by an example. Thus to find the logarithm of 1·085786 by the old tables we have $\log 1{\cdot}0857 = {\cdot}0357098$ and the proportional parts for 8 and 6, viz. 321, and 24 must be added to obtain $\log 1{\cdot}085786 = {\cdot}0357443$. From Mr. Sang's table we take out $1{\cdot}08578 = {\cdot}0357418$, and adding 25, the proportional part for 6, we obtain ·0357443, as before. It is

* Description des Grandes Tables logarithmiques et trigonométriques, par M. F. Lefort, t. IV., p. [123]. The tables are usually spoken of as the "grandes tables du Cadastre;" because they were calculated at the Bureau du Cadastre. A second copy of the Tables, which Prony retained, was presented by his heirs in 1858 to the Library of the Institute.

† Penny Cyclopœdia. Article, Prony.

also clear that, apart from the great saving of labour, Mr. Sang's table gives results of greater accuracy, as it requires the addition of only one proportional part instead of two. Anyone who will take out half-a-dozen logarithms from Mr. Sang's, and, say, Babbage's, near the beginning of the latter, will see how often the results differ by a unit in the last place, and this difference will generally be in favour of the former table. Another advantage, though one of far less importance than that just referred to, is that the sparseness of the difference-tables allows the multiples of the differences to be substituted for their proportional parts, which in some cases would give additional accuracy. With regard to the arrangement and printing of the tables we should expect a good deal from Mr. Sang, as besides being author of one of the clearest and most elegant tables published, viz. his Life Assurance and Annuity Tables,* he is well known as the editor of Major Shortrede's very elaborate tables. The present table is very clear and differs from all other tables we have ever seen in the fact of its being *entirely without rules*. With the exception of a thin black line running round the page and a similar line separating the logarithms from the difference-tables, the page has no lines whatever on it. This certainly lightens the appearance of the page, but one black line separating the numbers from the logarithms would, we think, have been an improvement. As it is, the logarithms are separated from the numbers by commas turned the wrong way. The figures which were supplied especially for the work are clear. We must own to preferring the printing of the numbers in a different type to the logarithms, as in Babbage, but this is a small matter. Of more importance is the manner of denoting a change in the fourth figure of the logarithm when it occurs in the middle of a block. Mr. Sang uses a black lozenge-shaped figure (resembling the diamond in a pack of cards) for 0 when the leading figure is changed. This method, though beautifully clear for denoting affected noughts, suggests no corresponding symbol for affected ones, twos, &c., which are therefore printed as usual. The small figures of Babbage, or the broken line of Callet, seem superior in this respect, as they call attention to all the affected figures. The method of prefixing an asterisk to all these figures as is done in Schrön's logarithms and in some modern French tables, seems much the best way of denoting the change, as by long practice an asterisk at once fixes the eye; and the late Prof. De Morgan has recorded his high opinion of this notation. There can, however, be no doubt that Mr. Sang has published unquestionably the best seven-place table of logarithms that has ever appeared, and no computer who has tried it will use any other. As the advantages are of such a kind that any one at all used to computing can detect them at once, the book must speedily secure universal adoption. It may be mentioned that the work—a large octavo—is not at all inconveniently large; it contains 365 pages, is less than twice as thick as Babbage, and very much thinner than either Hutton or Callet.

Mr. Sang announces that he proposes to publish by subscription a table of nine-place logarithms of all numbers from 100,000 to 1,000,000. The finished calculations are in a state of great forwardness, and the publication will proceed as soon as subscriptions are obtained sufficient to cover the outlay. Mr. Sang kindly forwarded, at our request, a specimen page of the table, which, besides the logarithms, contains the differences and multiples of the differences at the bottom of the pages. The work is to be published in three volumes of three parts each, price one guinea per part. The support of all interested in the advancement of exact science is earnestly requested for the undertaking, and we most sincerely trust it may obtain it, so that the publication of this magnificent table may not be delayed. The table will, it is almost needless to observe, be unique, as no million-table has ever been attempted before. A full account of Mr. Sang's method of calculation, which is briefly described in the preface to his seven-place table, has been presented to the Royal Society of Edinburgh. The method, however, is very similar to that adopted in the calculation of the French Tables du Cadastre.

J. W. L. GLAISHER.

---

* This work was set up by Mr. Sang's computers direct from the original calculations, to avoid the inaccuracies incidental to copying.

ON CERTAIN FAMILIES OF SURFACES.*

By *C. W. Merrifield, F.R.S.*

In a short note which I read before the Mathematical Society of London in the beginning of this year, I drew attention to an analytical distinction between conical and cylindrical surfaces, which appeared not to be sufficiently accounted for by the mere fact of the vertex moving off to infinity. As that note is very short, I reproduce it here.

If the equation of a surface be

$$z = F(x, y) \qquad (1),$$

it is very well known that the condition that it should be a ruled surface is that

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy}\right)^2 z \qquad (2)$$

and

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy}\right)^3 z \qquad (3)$$

should have a common factor of the form $A\lambda + B\mu$, and also that the condition of its being developable is that (2) should have two equal factors of that form.

I have found, upon actual trial, that for a conical surface (3) will have two equal factors, and for a cylindrical surface three equal factors, that is to say, if we write

$$\alpha = \frac{d^3z}{dx^3}, \quad \beta = \frac{d^3z}{dx^2dy}, \text{ \&c.},$$

we have, for a conical surface

$$(\alpha\delta - \beta\gamma)^2 = 4(\alpha\gamma - \beta^2)(\beta\delta - \gamma^2),$$

and for a cylindrical surface we have separately

$$\alpha\delta - \beta\gamma = 0, \quad \alpha\gamma - \beta^2 = 0, \quad \beta\delta - \gamma^2 = 0.$$

In fact, if we use the equation of a cone

$$z - c = (x - a)\, F\left(\frac{y - b}{x - a}\right),$$

* Read before the British Association Meeting, 1871.

we find, by differentiation,

$$\alpha\gamma - \beta^2 = -\frac{(y-b)^2}{(x-a)^6} F'''^2,$$

$$\beta\delta - \gamma^2 = -\frac{1}{(x-a)^4} F'''^2,$$

$$\alpha\delta - \beta\gamma = \frac{2(y-b)}{(x-a)^5} F'''^2,$$

whence $$(\alpha\delta - \beta\gamma)^2 = 4(\alpha\gamma - \beta^2)(\beta\delta - \gamma^2).$$

If we use the equation of a cylinder $ny - mz = F(mx - ly)$, we obtain

$$m\alpha = - m^3 F''',$$
$$m\beta = lm^2 F''',$$
$$m\gamma = - l^2 m F''',$$
$$m\delta = l^3 F''',$$

whence $$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy}\right)^3 z$$

takes the form $$\frac{F'''}{m}(l\mu - m\lambda)^3.$$

I had not then had time to look into the converse question as to whether the conditions of two and three equal roots really discriminated the cones and cylinders, that is to say, whether ruled surfaces, fulfilling these conditions, were necessarily cones or cylinders. A wider investigation has shewn me that I must look beyond ruled surfaces to ascertain the meaning of the invariant of the cubic.

Let a surface be generated by the motion of a plane parabola of the $(n-1)^{\text{th}}$ order, whose diameters always remain parallel to a fixed right line, which we will take for simplicity as parallel to the axis of z. The equations of the curve will then be

$$z = (*)(x, y, 1)^{n-1}, \quad y = cx + a.$$

And it is easily seen that

$$V_n = \left(\frac{d}{dx} + c\frac{d}{dy}\right) z = 0.$$

Moreover (see *Salmon's Higher Algebra*, 2nd edition, p. 100) the vanishing of the invariants of V_n, considered as a quantic in c, will not be affected by a linear change of coordinates in x, y, z.

For instance, if we have

$$z = lx + my + n, \quad y = cx + a,$$

we get, using the ordinary notation,

$$p + qc = l + mc,$$
$$r + 2sc + tc^2 = 0,$$

and the invariant $(rt - s^2)$ simply becomes $k(RT - S^2)$ if the coordinates are linearly changed.

The differential equation of the surface will in general be obtained by eliminating c between

$$V_n = \left(\frac{d}{dx} + c\frac{d}{dy}\right)^n z = 0,$$

and $$V_{n-1} = \left(\frac{d}{dx} + c\frac{d}{dy}\right)^{n+1} z = 0.$$

But if the generating curve has an envelope, we may eliminate c between $V_n = 0$ and $\frac{dV_n}{dc} = 0$. That is to say, $V_n = 0$ will have a pair of equal roots, considered as a quantic in c. If, again, the generating curve meets the envelope in three consecutive points, or, what is the same, if two branches of the envelope coincide, $V_n = 0$ will have three equal roots.

Take for instance the plane parabola $4ay = x^2$; the equation of its circle of curvature is

$$v = 2a^2(\xi^2 + \eta^2) + \xi x^3 - a\eta(8a^2 + 3x^2) - \frac{3x^4}{8} = 0,$$

$$\frac{dv}{dx} = 3\xi x^2 - 6a\eta x - \frac{3}{2}x^3 = 0.$$

The result of eliminating x between these is

$$(\xi^2 - 4a\eta)^2(a^2 + \xi^2 - 2a\eta + \eta^2) = 0.$$

The last factor, being the sum of two squares, is extraneous* to the problem, and there remains the original parabola *as a double line.* It is also easily seen that eliminating x between $\frac{dv}{dx} = 0$ and $\frac{d^2v}{dx^2} = 0$ gives $\xi^2 = 4a\eta$. The geometry of this is evident: a family of circles will in general have two

* It has since been pointed out to me that this factor, instead of being extraneous, gives the focus, which has double contact (imaginary) with the parabola.

envelopes, and the consecutive circles will cut in two points; but when they osculate the envelope, the successive circles touch one another, and the two envelopes coincide.

When, however, the curves are in space, instead of in a plane, the conditions are different. Circles, for instance, may meet once and not again. Thus if a vertical circle has its centre on a helix, and always touches the axis (supposed vertical) the axis is the simple envelope of the family of circles. But if a moving circle always osculates the helix, it must fulfil the two conditions of having the same osculating plane, and the same absolute curvature. There will then be three points in common, or, algebraically,

$$v = 0, \quad \frac{dv}{dc} = 0, \quad \frac{d^2v}{dc^2} = 0.$$

In the particular case which we began by considering, the plane of the generator was parallel to a fixed line. Hence an envelope osculated by the generator must have its osculating plane parallel to a fixed line. It must therefore be a plane curve.

This is geometrically evident, but it may easily be shown algebraically as follows:—Let the direction cosines of the line be lmn and let

$$D = \begin{vmatrix} dx, & dy, & dz \\ d^2x, & d^2y, & d^2z \\ d^3x, & d^3y, & d^3z \end{vmatrix}.$$

Then the condition that the osculating plane should be parallel to the given line is

$$l\{dy\,d^2z - d^2y\,dz\} + m\{dz\,d^2x - d^2z\,dx\} + n\{dx\,d^2y - d^2x\,dy\} = 0.$$

And this is to hold true at all points of the curve. We may therefore differentiate it, and we get

$$l\{dy\,d^3z - d^3y\,dz\} + m\{\ldots = 0,$$

whence, by a well known reduction, we obtain

$$l : m : n :: Ddx : Ddy : Ddz.$$

And since $l : m : n$ are fixed ratios, the curve will degenerate into a straight line, unless $D = 0$, in which case it is a plane curve.

Let us now consider the surface of which the generator is a common parabola, variable in size, moving with its

diameter always parallel to a fixed line, which we shall take for the axis of z. We shall have

$$V_3 = \left(\frac{d}{dx} + c\frac{d}{dy}\right)^3 z = \alpha + 3\beta c + 3\gamma c^2 + \delta c^3 = 0,$$

and the general equation of the surface will be got by eliminating c between this and $V_4 = 0$.

If the successive generators intersect, we shall have equal roots in the cubic, and therefore

$$(\beta\gamma - \alpha\delta)^2 = 4(\beta^2 - \alpha\gamma)(\gamma^2 - \beta\delta).$$

Besides the plane, we get two marked cases of singularity.

1. Where the envelope degenerates into a point; that is to say, where a variable parabola revolves round a diameter, and passes through a fixed point. We get, as a particular case, the spindle made by causing a fixed parabola to revolve round any diameter, preserving the vertex. As cases of different degradations, we get the paraboloid and conical surfaces.

2. Where the envelope moves off to infinity, we then get the ruled surface with a director plane.

If the generators osculate their envelope, we have three equal roots to the equation

$$\alpha + 3\beta c + 3\gamma c^2 + \delta c^3 = 0.$$

The conditions for these are

$$\beta^2 - \alpha\gamma = 0, \quad \gamma^2 - \beta\delta = 0, \quad \text{and therefore } \beta\gamma - \alpha\delta = 0.$$

The envelope now becomes a plane curve, and the surface degenerates into a plane, *unless* the osculation takes place at infinity. This exception includes all cylindrical surfaces, and cylindrical surfaces only.

In the case of a parabolic spindle on a diameter,

$$2z = x^2 + y^2 + a^2 + 2a\sqrt{(x^2 + y^2)},$$

we have
$$\alpha = -\frac{3axy^2}{(x^2+y^2)^{\frac{5}{2}}}, \quad \beta = \frac{ay(2x^2 - y^2)}{(x^2+y^2)^{\frac{5}{2}}},$$

$$\delta = -\frac{3ax^2y}{(x^2+y^2)^{\frac{5}{2}}}, \quad \gamma = \frac{ax(2y^2 - x^2)}{(x^2+y^2)^{\frac{5}{2}}},$$

or, neglecting the common denominator,

$$\beta^2 - \alpha\gamma = y^2(x^2+y^2)^2,$$
$$\gamma^2 - \beta\delta = x^2(x^2+y^2)^2,$$
$$\alpha\delta - \beta\gamma = 2xy(x^2+y^2)^2,$$

therefore $$(\beta\gamma - \alpha\delta)^2 = 4(\beta^2 - \alpha\gamma)(\gamma^2 - \beta\delta).$$

This verifies the theorem. I have already shown that among ruled surfaces, the cone fulfils this condition; and it is easy to show that, if we take the equation of a ruled surface with a director plane parallel to z, namely,

$$z = x\phi(ax + by) + y\psi(ax + by) + \chi(ax + by),$$

we get a similar result; in fact, this leads to (*see* MONGE)

$$b^2r - 2abs + a^2t = 0,$$

and since b and a are constants, and this equation holds for the whole surface, we may differentiate partially with respect to x and y separately, which gives

$$b^2\alpha - 2ab\beta + a^2\gamma = 0,$$

$$b^2\beta - 2ab\lambda + a^2\delta = 0;$$

and, eliminating the ratio $a : b$, we obtain the condition that the cubic discriminant shall vanish.

The connecting link between these two families of surfaces is the cylinder. The reasoning which I have used appears to show that it is only cylindrical surfaces which fulfil the condition that there should be three equal roots in the equation $V_n = 0$, not only among ruled surfaces, but also among those traced out by a plane parabola of any order whatever moving with its diameter parallel to a fixed line. It thus becomes unnecessary to enquire into the conditions which distinguish the cases in which there are more than three equal roots in $V_n = 0$. At any rate, this point would require an analysis altogether different. I have no doubt restricted my solution by using, as one of the equations of the generator, $y = cx + a$, and by making the coefficient of x in that equation the parameter with respect to which we take the envelope. But this restriction was necessary to my method of investigation.

It may be worth while again to remark that the invariance of the conditions allows of any linear change in the coordinates.

The general equation of the family of surfaces traced out by the curve

$$z = (*)(x, y, 1)^{n-1}, \quad y = cx + a$$

appears, like that of ruled or developable surfaces in general, incapable of a functional expression. There is no difficulty in expressing it when sufficiently conditioned, as for instance

when the parabolic generator is parallel to a director plane. Assuming the director plane to be that of (y, z) we simply require that $x = c$ should give

$$z = (*)\,(y, 1)^{n-1},$$

and for this purpose we have only to replace the constants of the latter equation by arbitrary functions of x. We can afterwards, if we think fit, introduce a linear change of coordinates.

The classification of ruled surfaces here indicated has been obtained by the consideration of surfaces generated by *plane parabolas with diameters parallel to a fixed line.* The restriction introduced by these words runs throughout the investigation.

To sum up: the condition of three equal roots in the cubic $V_3 = 0$ discriminates completely the cylinders; but the condition of two equal roots does not discriminate the cones from other developable surfaces. In fact, by differentiating $rt - s^2$, it is easy to show that if $V_2 = 0$ has a pair of equal roots, each of the other V's will also contain a pair of equal roots.

Royal School of Naval Architecture,
South Kensington Museum,
June, 1871.

EXTRACT FROM A LETTER FROM PROF. CAYLEY TO MR. C. W. MERRIFIELD.

The general integral of the equations

$$\frac{\alpha}{\beta} = \frac{\beta}{\gamma} = \frac{\gamma}{\delta},$$

can, I think, be found, viz. $\frac{\alpha}{\beta} = \frac{\beta}{\gamma}$ gives $r =$ function s, and $\frac{\beta}{\gamma} = \frac{\gamma}{\delta}$ gives $s =$ function t. But $r =$ function s is integrated as the equation of a developable surface (p instead of z), viz. we have

$$\left.\begin{aligned} p &= ax + hy + g \\ 0 &= a'x + y + g' \end{aligned}\right\} \quad a \text{ and } g \text{ functions of } h, \text{ and } \left(a' = \frac{da}{dh},\ g' = \frac{dg}{dh}\right),$$

similarly, $s=$ function t, gives

$$q=hx+by+f,$$
$$0=x+b'y+f',\ \left(b'=\frac{db}{dh},\ f'=\frac{df}{dh}\right),$$

observe that the constants have been so taken, that $\frac{dp}{dy}=h$, $\frac{dq}{dx}=h$; but in order that h may, in the two pairs of equations, mean the same function of (x, y), we must have

$$a'=\frac{1}{b'}=\frac{g'}{f'},$$

that is
$$b=\int\frac{dh}{a'},\ f=\int\frac{g'dh}{a'},$$

or, writing $a=\phi h$, $g=\chi h$, we have

$$p=x\phi h+yh+\chi h,$$
$$q=hx+y\int\frac{dh}{\phi' h}+\int\frac{\chi' h.dh}{\phi' h},$$

where
$$x\phi' h+y+\chi' h=0.$$

The last equation gives h as a function of (x, y), and the values of p, q are then such that $dz=pdx+qdy$ is a complete differential, so that we obtain z by the integration of that equation.

A simple example is

$$p=\tfrac{1}{2}h^2x-hq,\ q=-hx+y\log h,\ hx-y=0,$$

that is
$$p=-\frac{1}{2}\frac{y^2}{x},\ q=-y+y\log\frac{y}{x},$$

whence
$$z=\tfrac{1}{2}y^2\log\frac{y}{x}-\tfrac{3}{4}y^2,$$

we have
$$r-\frac{1}{2}\frac{y^2}{x^2},\ s=-\frac{y}{x},\ t=\log\frac{y}{x},$$
$$\alpha=-\frac{y^2}{x^3},\ \beta=\frac{y}{x^3},\ \gamma=-\frac{1}{x},\ \delta=\frac{1}{y},$$

or
$$\frac{\alpha}{\beta}=\frac{\beta}{\gamma}=\frac{\gamma}{\delta}\left(=-\frac{y}{x}\right)\text{ as it should be.}$$

Cambridge, 28 *July*, 1871.

SOLUTIONS OF A SMITH'S PRIZE PAPER FOR 1871.

(Continued from p. 77).

By Professor *Cayley*.

10. *An endless heavy chain of given length is suspended from two fixed points in the same horizontal plane: shew that (subject to a condition as to the length) the figure of equilibrium may consist of portions of two distinct catenaries.*

The two parts of the chain will each of them be a portion of a catenary, viz. they will either coincide with each other, forming a twice repeated portion of a catenary (which is always a possible position of equilibrium), or they will form portions of two distinct catenaries. That the latter form is in some cases possible, appears from the case of a very long chain. It is then clear that there is a position of equilibrium in which the upper catenary is nearly a straight line. It may be added, that, as the length of the chain diminishes, the two distinct catenaries approach more and more, and for a certain value of the length become coincident; for any smaller value of the length, the only position is that consisting of a twice repeated portion of a catenary. But to obtain the solution in a regular manner, observe that, in order to the existence of such a form of equilibrium, the necessary condition is, that the tension at A (or B) must be equal in the two catenaries. Now the tension at any point of a catenary is proportional to the height above the directrix of the catenary; hence the condition, that there shall be through the points A, B two catenaries having the same directrix, and such that the sum of the lengths is equal to the given length of the chain.

Take $AB = 2a$, the length of the chain $= 2l$. Take β for the distance of the directrix below the points A, B; c for the parameter of the catenary (or distance of its lowest point above the directrix, β, c being of course unknown. Then taking the origin at the mid-point of the directrix, and the axis of y vertically upwards, the equation of the catenary is

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right),$$

whence for the point A or B,

$$\beta = \frac{c}{2}\left(e^{\frac{a}{c}} + c^{-\frac{a}{c}}\right),$$

and the arc measured from the lowest point is

$$s=\frac{c}{2}\left(e^{-\frac{x}{c}}-e^{-\frac{x}{c}}\right).$$

Hence, assuming that there are two distinct catenaries, if the parameters are c, c', we have

$$\tfrac{1}{2}c\left(e^{\frac{a}{c}}+e^{-\frac{a}{c}}\right)=\tfrac{1}{2}c'\left(e^{\frac{a}{c'}}+e^{-\frac{a}{c'}}\right),$$

$$\tfrac{1}{2}c\left(e^{\frac{a}{c}}-e^{-\frac{a}{c}}\right)+\tfrac{1}{2}c'\left(e^{\frac{a}{c'}}-e^{-\frac{a}{c'}}\right)=l,$$

which are the conditions for the determination of c, c'; and it is to be shewn that these can be satisfied otherwise than by taking $c=c'$.

Trace the two curves

$$y=\frac{x}{2}\left(e^{\frac{a}{x}}+e^{-\frac{a}{x}}\right),$$

$$y'=\frac{x}{2}\left(e^{\frac{a}{x}}-e^{-\frac{a}{x}}\right),$$

shewn respectively by the black line and the dotted line in fig. 9. Draw any line parallel to the axis of x, meeting the first curve in the points P, P' respectively, and let the ordinates MP, $M'P'$ meet the second curve in the points Q, Q' respectively; then it is clear, that if for a given value of l the line PP' can be drawn in suchwise that $MQ+M'Q'=l$, there will be in fact the required two values $C=OM$ and $C'=OM'$

And since for MP very large we have MQ, and therefore also $MQ+M'Q'$ very large, and as MP diminishes, $MQ+M'Q'$ also diminishes until it attains a certain minimum value, say $=\lambda$, it is clear that if l has any value greater than this minimum value, PP' can be so drawn that $QM+Q'M'=l$.

[The above remarkably elegant investigation in regard to the two values C, C' was given in the Examination; it seems to be the case that as PP' moves downwards, $MQ+M'Q'$ continually decreases (viz. MQ decreases more rapidly than $M'Q'$ increases), its value being least, and $=2NS$ when PP' becomes a tangent to the first curve at its lowest point R; but it is not by any means easy to prove that this is so. The question depends on the form of the curve defined by the equations

$$X=\tfrac{1}{2}x_1\left(e^{\frac{a}{x_1}}-e^{-\frac{a}{x_1}}\right)+\tfrac{1}{2}x_2\left(e^{\frac{a}{x_2}}-e^{-\frac{a}{x_2}}\right),$$

$$Y=\tfrac{1}{2}x_1\left(e^{\frac{a}{x_2}}+e^{-\frac{a}{x_2}}\right)=\tfrac{1}{2}x_2\left(e^{\frac{a}{x_2}}+e^{-\frac{a}{x_2}}\right),$$

where X and Y are the current coordinates].

11. *A particle is attracted to two centres of force, one of them at the origin, the other revolving about the origin in a circle in the plane of xy with a uniform angular velocity n′: find the equations of motion; and writing v for the velocity of the particle and A for the resolved area (about the fixed centre) in the plane of xy, shew that there is a first integral giving the value of* $v^2 - 4n'\dfrac{dA}{dt}$ *in terms of the coordinates of the particle and of the revolving centre.*

Take ξ, η, 0 for the coordinates of the moving centre

$$\xi = a\cos n't, \quad \eta = a\sin n't,$$

the equations of motion are

$$\frac{d^2x}{dt^2} = -\phi r\frac{x}{r} - \psi\rho\frac{x-\xi}{\rho},$$

$$\frac{d^2y}{dt^2} = -\phi r\frac{y}{r} - \psi\rho\frac{y-\eta}{\rho},$$

$$\frac{d^2z}{dt^2} = -\phi r\frac{z}{r} - \psi r\ \frac{z}{\rho}\ .$$

where $\quad r^2 = x^2 + y^2 + z^2,$

$$\rho^2 = (x-\xi)^2 + (y-y)^2 + z^2,$$

we have $\quad r\,dr = x\,dx + y\,dy + z\,dz,$

$$\rho\,d\rho = (x-\xi)(dx - d\xi) + (y-\eta)(dy - d\eta) + z\,\delta z.$$

But $\quad d\xi = -n'a\sin n't\,dt = -n'\eta\,dt,$

$$d\eta = \ \ n'a\cos n't\,dt = \ \ n'\xi\,dt,$$

whence

$$\begin{aligned}\rho\,d\rho &= (x-\xi)\,dx + (y-\eta)\,dy + z\,dz \\ &\quad + n'[\eta(x-\xi) - \xi(y-\eta)]\,dt \\ &= (x-\xi)\,dx + (y-\eta)\,dy + z\,dz \\ &\quad - n'[x(y-\eta) - y(x-\xi)]\,dt.\end{aligned}$$

Hence from the equations of motion

$$2\left(\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}\right)$$

$$-2n'\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right)$$

$$=-\frac{\phi r}{r}\,2\left(x\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$
$$-\frac{\psi\rho}{\rho}\left[2\left\{(x-\xi)\frac{dx}{dt}+(y-\eta)\frac{dy}{dt}+z\frac{dz}{dt}\right]\right.$$
$$\left.-2n'\left\{x(y-\eta)-y(x-\xi)\right\}\right].$$

But we have

$$v^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right),$$

$$x\frac{d^2y}{dt}-y\frac{d^2x}{dt^2}=\frac{d}{dt}r^2\frac{d\theta}{dt}=2\frac{dA}{dt},$$

and the equation may be written

$$\frac{d.v^2}{dt}-4n'\frac{d^2A}{dt^2}=-2\phi r\frac{dr}{dt}-2\psi\rho\frac{d\rho}{dt},$$

whence $\quad v^2-4n'\dfrac{dA}{dt}=C-2\int\phi r\,dr-2\int\psi\rho\,d\rho,$

the required result.

12. *If x, y are the coordinates of a particle moving* in plane *under the action of a central force varying as (distance)$^{-2}$: write down the expressions of the coordinates x, y in terms of the time t and of four arbitrary constants: and (in case of disturbed motion) starting from the equations*

$$\delta x=0,\quad \delta y=0,\quad \delta x'=\frac{d\Omega}{dx}dt,\quad \delta y'=\frac{d\Omega}{dy}dt,$$

(the notation to be explained) indicate the process of finding the variations of the constants in terms of (1) $\dfrac{d\Omega}{dx}$, $\dfrac{d\Omega}{dy}$, (2) *the derived functions of Ω in regard to the constants.*

We have

$$x=a\left\{\frac{\cos u-e}{1-e\cos u}\cos\varpi+\frac{\sqrt{(1-e^2)}\sin u}{1-e\cos u}\sin\varpi\right\},$$

$$y=a\left\{\frac{\sqrt{(1-e^2)}\sin u}{1-e\cos u}\cos\varpi-\frac{\cos u-e}{1-e\cos u}\sin\varpi\right\},$$

where $\qquad n-e\sin u=t\sqrt{\left(\dfrac{\mu}{a^3}\right)}+c,$

an equation serving to express u in terms of t and the

constants a, e, c; the foregoing equations, therefore, in effect give x, y in terms of t and the four constants a, e, c, ϖ.

In the second part of the question Ω is a given function of x, y, t, the differential coefficients $\frac{d\Omega}{dx}$, $\frac{d\Omega}{dy}$ being the partial ones in regard to x, y respectively. The equation $\delta x = 0$ signifies that the variation of x, in so far as it arises from the variation of the constants, is $=0$, it in fact *means*

$$\frac{dx}{da}\frac{da}{dt}+\frac{dx}{de}\frac{de}{dt}+\frac{dx}{dc}\frac{dc}{dt}+\frac{dx}{d\varpi}\cdot\frac{d\varpi}{dt}=0.$$

The value of $x'\left(=\frac{dx}{dt}\right)$ is therefore obtained from that of x by differentiating in regard to t alone, as if a, e, c, ϖ were constants: viz. x' will be a given function of t, a, e, c, ϖ; $\delta x'$ then denotes the variation of x' in so far as it arises from the variation of the constants: viz. the equation $\delta x' = \frac{d\Omega}{da}\,dt$ *means*

$$\frac{dx'}{da}\frac{da}{dt}+\frac{dx'}{de}\frac{de}{dt}+\frac{dx'}{dc}\frac{dc}{dt}+\frac{dx'}{d\varpi}\cdot\frac{d\varpi}{dt}=\frac{d\Omega}{dx}.$$

There are the like equations in regard to y, y', viz. in all four equations linear in regard to $\frac{da}{dt}$, $\frac{de}{dt}$, $\frac{dc}{dt}$, $\frac{d\varpi}{dt}$; and which serve to determine these quantities in terms of $\frac{d\Omega}{dx}$, $\frac{d\Omega}{dy}$.

Now considering the x, y as expressed in terms of a, e, c, ϖ, t then Ω becomes a function of these quantities; the differential coefficients $\frac{d\Omega}{da}$, &c., being connected with the original differential coefficients $\frac{d\Omega}{dx}$, $\frac{d\Omega}{dy}$ by the equations

$$\frac{d\Omega}{da}=\frac{d\Omega}{dx}\frac{dx}{da}+\frac{d\Omega}{dy}\frac{dy}{da},$$

$$\frac{d\Omega}{de}=\frac{d\Omega}{dx}\frac{dx}{de}+\frac{d\Omega}{dy}\frac{dy}{de},$$

&c.

As there are four equations $\frac{d\Omega}{dx}$, $\frac{d\Omega}{dy}$ can be expressed in an infinity of ways in terms of $\frac{d\Omega}{da}$, $\frac{d\Omega}{de}$, $\frac{d\Omega}{dc}$, $\frac{d\Omega}{d\varpi}$, and

considering $\frac{da}{dt}$, &c., as given in terms of $\frac{d\Omega}{dx}$, $\frac{d\Omega}{dy}$, we can in an infinity of ways express $\frac{da}{dt}$, &c., as linear functions of $\frac{d\Omega}{da}$, $\frac{d\Omega}{de}$, $\frac{d\Omega}{dc}$, $\frac{d\Omega}{d\varpi}$. But there is one form (obtained by combining the equations in a particular manner) wherein the coefficients of the last-mentioned quantities are functions of a, e, c, ϖ *without* t; and this is the form actually employed for the expression of $\frac{da}{dt}$, $\frac{de}{dt}$, $\frac{dc}{dt}$, $\frac{d\varpi}{dt}$ in terms of $\frac{d\Omega}{da}$, $\frac{d\Omega}{de}$, $\frac{d\Omega}{dc}$, $\frac{d\Omega}{d\varpi}$, in the method wherein these quantities are made use of.

I remark upon the present question, that the answer *ought* to be *in substance perfectly familiar to every student in Physical Astronomy;* and that a student *ought* to be able to present it *in a clear and logical form*: the question being in fact intended as a test of ability *in this respect.*

13. *Explain the course of the geodesic lines on a spheroid of revolution: and in particular shew that the condition is satisfied in virtue of which any geodesic line, considered as starting from a given point, ceases at some point of its course to be a shortest line.*

From each point on the surface a geodesic line may be drawn in any direction whatever along the surface, that is, through each point of the surface there is a singly infinite series of geodesic lines. A geodesic line undulates (in the manner of a sinusoid) between two parallels equidistant from the equator on opposite sides thereof; viz. considering it as starting from a point A on the equator, it arrives at a point V on the upper parallel (there touching the parallel), and passes downwards to cut the equator at A', and thence arrives at a point V' on the lower parallel (there touching the parallel), and again passes upwards to meet the equator at A'', and so on; the arcs AV, VA', $A'V'$, $V'A''$, &c., being similar and equal to each other (differing only in position). The equatoreal arc AA' $(=A'A''=\&c.)$ or difference of the longitudes A, A', is always less than 180°, its value increasing with the inclination at which the geodesic line cuts the equator [viz. when this angle is indefinitely small, the arc is $=\frac{c}{a}180°$ (c, a the polar and equatoreal axes respectively), and as the inclination becomes indefinitely near to 90°, the value

of the arc becomes indefinitely near 180°]. If the arc in question is commensurable with 180°, the geodesic line will be, it is clear, a closed curve, but if not, then it is not a closed curve, but proceeds undulating for ever between the two parallels. In the limiting case where the inclination is $= 90°$, the geodesic line is obviously a meridian.

Considering a geodesic line starting in a given direction from a point A, and the geodesic line from the same point A in the consecutive direction, it appears from the foregoing account of the configuration of the lines, that the two lines will intersect each other in general an indefinite number of times: supposing that they first intersect in a point K, then by a general theorem of Jacobi's, the geodesic line AK is a shortest line from A to any point nearer than K, but it is not a shortest line from A to any point beyond K.

THE BRITISH ASSOCIATION MEETING AT EDINBURGH.

The forty-first meeting of the British Association for the Advancement of Science was opened at Edinburgh on the evening of Wednesday, August 2, by the address of the President, Sir W. Thomson, LL.D., F.R.S., &c. In the course of his discourse he paid a high tribute to the memory of A. De Morgan and Sir John Herschel; the latter, he remarked, had done more to introduce into Britain the powerful methods and the valuable notation of modern analysis than any other man. The separation of symbols was with the French writers rather a short method of writing formulæ than the analytical engine it became in the hands of Herschel and British followers, especially Sylvester, Gregory, Boole, and Cayley. The method was greatly advanced by Gregory, who first gave to its working power a secure and philosophical foundation, and so prepared the way for the marvellous extension it has received from Boole, Sylvester, and Cayley, according to which symbols of operation become the subjects not merely of algebraical combinations, but of differentiations and integrations, as if they were symbols expressing values of varying quantities. An even more marvellous development of the same idea of the separation of symbols received from Hamilton a most astonishing generalization, by the invention actually of new laws of combination, and led him to his famous Quaternions. The subject had been taken up by Tait and carried into physical science with a faith, shared by some of the most thoughtful naturalists of the day, that it is destined to become an engine of perhaps hitherto unimagined power for investigating and expressing results in Natural Philosophy. The President further on referred in high terms to Cayley's report on Abstract Dynamics, and Maxwell's researches on the Kinetic Theory of Gases.

The Sections assembled on Thursday morning, August 3, Section A (Mathematical and Physical Science), being presided over by Prof. Tait, who delivered the inaugural address, which dealt chiefly with the subjects of Quaternions and dissipation of energy.

The great and probably unprecedented number of mathematicians present at the meeting excited general attention. The Committee of Section A comprised the names of Prof Cayley, Prof. Stokes, Prof. Sylvester, Sir W. Thomson, Prof. Clerk Maxwell, Dr. Spottiswoode, Prof. Challis, Prof. Adams, Prof. Rankine, Rev. R. Harley, Mr. C. W. Merrifield, Mr. W. H. L. Russell, Prof. Ball, Prof. Clifford, &c. A paper read by Dr. Carpenter "On the Thermo-dynamics of the General Oceanic Circulation" led to an interesting discussion on the effect of wind on the motion of waves and currents in the open sea and in friths, in which Prof. Stokes and Sir W. Thomson took part (this discussion is printed in *Nature* for August 17).

On Friday a paper was read by Prof. Stokes, entitled "Notice of the Researches of the late Rev. W. V. Harcourt on the Conditions of Transparency of Glass," the results of Mr. Harcourt's elaborate researches having resulted in the preparation of discs of terborate of lead and of a titanic glass, of about 3 inches diameter, almost homogeneous, and with which it is intended to attempt the construction of an actual object glass, which shall give images free from secondary colour.

Saturday was, as usual, devoted by the Section almost exclusively to pure mathematical subjects. The following papers were read: "Report of the Tidal Committee," by Sir W. Thomson. "Report on Hyperelliptic Functions," "On Focal Properties of Surfaces of the Second Order," and "On Linear Differential Equations," by W. H. L. Russell, F.R.S. On the subject of the last paper see *Proc. Roy. Soc.*, vol. XIX, p. 526. "Note on a Question of Partitions," by Prof. Sylvester. "On the Number of Invariants of a Binary Quartic," by Prof. Cayley. "On certain Families of Surfaces," by C. W. Merrifield, F.R.S. [This paper is published in the present number of the *Messenger*.] "On Vortex Rings" and "Description of a Model of a Ruled Cubic Surface," by Prof. Ball. The model, which represented the surface $z(x^2+y^2)-2axy=0$, was exhibited. "On the Mathematical Theory of Atmospheric Tides," by Prof. Challis. "Remarks on Napier's Original Method of Logarithms," by Prof. Purser. "On the Calculation of e from a Continued Fraction," by J. W. L. Glaisher. The continued fraction from which e was calculated was $\dfrac{e-1}{2}=\dfrac{1}{1+}\,\dfrac{1}{6+}\,\dfrac{1}{10+\&c.}$, a formula far more convergent than the ordinary one for e in a series. The calculation gave e to 137 places, and confirmed the result given by Mr. Shanks; the value of e given in all the editions of Callet's logarithms being incorrect from the fortieth figure. "On Certain Definite Integrals," by J. W. L. Glaisher. [A portion of this paper will appear in a future number of the *Messenger*.] "On Lambert's Proof of the Irrationality of π and on the Irrationality of Certain Other Quantities," by J. W. L. Glaisher. The quantities referred to were chiefly circular and exponental functions. "On Doubly Diametrical Quartan Curves," by Prof. F. W. Newman. A large number of drawings of quartic curves were exhibited to the Section. "On a Canonical Form of Spherical Harmonics," by Prof. Clifford. The canonical form in question is an expression of the general harmonic of order n as the sum of a certain number of sectorial harmonics, the number being $\dfrac{5n-10}{2}$ when n is even, and $\dfrac{5n-9}{2}$ when n is odd. Papers were also subsequently read by Sir W. Thomson "On a General Canonical Form of a Spherical Harmonic of the n^{th} Order," and by Prof. Clifford "Note on the Secular Cooling of the Earth." In the latter paper Prof. Clifford stated that, assuming (in accordance with Dr. Calvert's views) that the initial temperature of the earth was 320° F., the increase of time required was only 8 per cent. as compared with the initial temperature of 50°.

The following Committees on mathematical subjects were appointed by the General Committee, having been previously recommended by the Committee of Section A:

(1) That a Committee, consisting of Prof. Cayley, Mr J. W. L. Glaisher, Prof. H. J. S. Smith, Prof. Stokes, and Sir W. Thomson be appointed to prepare a catalogue of mathematical tables; to contain an Account of all numerical tables in algebra, finite differences and theory of numbers, as well as tables of transcendental functions (exclusive of ordinary logarithmic and trigonometrical tables); and also to reprint or compute such tables as seem desirable. That Mr. J. W. L. Glaisher be the Secretary; and that the sum of £50 be placed at the disposal of the Committee for the purpose.

(2) That the Committee on the Tides be re-appointed; and that the sum of £200 be placed at their disposal for the purpose.

(3) That Mr. W. H. L. Russell be requested to continue his Report on Hyperelliptic Functions.

(4) That Prof. Tait be requested to Report on the Application of Quaternions to Physical Problems.

The initiative in a matter of great importance to the whole of the science was also taken in the following recommendation: "That the Council be requested to take into consideration the desirability of the publication of periodic records of the advances made in the various branches of science represented by the British Association." It was decided also that application be made to the Treasury for funds to enable the Tidal Committee to continue their calculations.

ON A PARADOX IN INFINITE SERIES.

By *J. W. L. Glaisher, B.A.*

THE paradox is as follows:

Let $$S = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots \qquad \dots\dots\dots\dots (1),$$

and $$\alpha = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots \qquad \dots\dots\dots\dots (2),$$

then, since the terms in (2) are alternately positive and negative, and continually decrease, α must lie between $\frac{1}{\sqrt{2}}$ and 1, and must therefore be finite and positive.

Subtract (2) from (1) and we have

$$S - \alpha = 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}} + \dots\right) \dots\dots\dots\dots (3),$$

$$= \sqrt{2}\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots\right) = \sqrt{2} . S,$$

whence $$S(\sqrt{2} - 1) = -\alpha, \text{ or } S = -\frac{\alpha}{\sqrt{2} - 1},$$

that is a positive quantity—for from (1) S is clearly positive—is equal to a negative quantity. The value of S is known to be infinite, for each term of (1) is greater than the corresponding term of the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \dots$, so that the paradox is that a quantity, positive and infinite, is equal to a finite negative quantity.

It should be noticed that the difficulty is of a different nature from the result obtained by making $x = 2$ in the series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \qquad \dots\dots\dots\dots (4),$$

viz. $$-1 = 1 + 2 + 2^2 + 2^3 + \dots,$$

for in obtaining (4) by division we neglect an infinite negative remainder.

The explanation of the paradox is easily seen to depend upon the fact, that although the number of terms both in (1) and (3) is infinite, the infinity in the former series is double that of the latter; thus, let

$$S_{2n} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \dots + \frac{1}{\sqrt{(2n)}},$$

then $$S_{2n} - \alpha = 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \dots + \frac{1}{\sqrt{(2n)}}\right)$$

$$= \sqrt{2}\left(1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}\right) = \sqrt{2} \, . \, S_n \dots\dots (5).$$

The error consists in assuming that when n is infinite $S_{2n} = S_n$; knowing that they are infinite, the above investigation shews that S_{2n} is to S_n as $\sqrt{2}$ to 1. This can easily be verified, for from Bernoulli's series, we have

$$1 + \frac{1}{\sqrt{2}} \dots + \frac{1}{\sqrt{n}} = C + \int \frac{dn}{\sqrt{n}} + \frac{1}{2\sqrt{n}} + \frac{B_1}{\lfloor 2} \frac{d}{dn} \frac{1}{\sqrt{n}} - \frac{B_2}{\lfloor 4} \frac{d^3}{dn^3} \frac{1}{\sqrt{n}} + \dots$$

$$= C + 2\sqrt{n} + \frac{1}{2\sqrt{n}} - \frac{1}{24 n^{\frac{3}{2}}} + \frac{1}{384 n^{\frac{7}{2}}} - \dots\dots\dots (6),$$

whence, if n is infinite $\dfrac{S_{2n}}{S_n} = \sqrt{2}$. By comparing (5) and (6) we can also determine C, viz. $C(1 - \sqrt{2}) = \alpha$, so that

$$C = -\frac{1}{\sqrt{2} - 1}\left(1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \dots\right),$$

which, although not a good formula to calculate C from, is convergent while any series obtained by giving n a particular value in (6) would be only semi-convergent.

It can be shewn that $S_{2n} - S_n$ is infinite without having recourse to Bernoulli's series, for

$$S_{2n} - S_n = \frac{1}{\sqrt{(n+1)}} + \frac{1}{\sqrt{(n+2)}} \dots + \frac{1}{\sqrt{(2n)}},$$

which is obviously greater than $\dfrac{n}{\sqrt{(2n)}}$ or $\sqrt{\dfrac{n}{2}}$.

Its exact value (n being infinite)

$$=\sqrt{n}.\frac{1}{n}\left\{\frac{1}{\sqrt{\left(1+\frac{1}{n}\right)}}+\frac{1}{\sqrt{\left(1+\frac{2}{n}\right)}}\ldots+\frac{1}{\sqrt{\left(1+\frac{n}{n}\right)}}\right\}$$

$$=\sqrt{n}\int_0^1\frac{dx}{\sqrt{(1+x)}}=2\sqrt{(2n)}-2\sqrt{n},$$

which agrees with (6).

A similar treatment of the harmonic series $1+\frac{1}{2}\ldots+\frac{1}{x}$ affords results of interest.

The following has been given as a proof that the harmonic series is divergent.

Let $$s=1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots \qquad (7),$$

$$\beta=1-\tfrac{1}{2}+\tfrac{1}{3}-\ldots \qquad (8),$$

whence β lies between $\frac{1}{2}$ and 1, and must be finite.

By subtraction

$$s-\beta=2\left(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6}+\ldots\right)=s \qquad (9),$$

and therefore s must be infinite.

The correct result in place of (9), is

$$s_{2n}-\beta=s_n, \text{ or } s_{2n}-s_n=\beta.$$

Now $$s_{2n}-s_n=\frac{1}{n+1}+\frac{1}{n+2}\ldots+\frac{1}{2n}$$

$$=\frac{1}{n}\left\{\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}\ldots+\frac{1}{1+\frac{n}{n}}\right\}$$

$$=\int_0^1\frac{dx}{1+x}=\log 2,$$

so that we obtain $\beta=\log 2$,* the truth of which is seen by putting $x=1$ in the series

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots.$$

---

* Euler in t. VII. of the *Comm. Acad. Petropol.*, finds that if n be infinite, $1+\frac{1}{2}\ldots+\frac{1}{n}=\log\frac{1}{0}$, $\frac{1}{n}+\frac{1}{n+1}\ldots+\frac{1}{2n}=\log\frac{2}{1}$, $\frac{1}{2n}+\ldots+\frac{1}{3n}=\log\frac{3}{2}$, &c., a result which he speaks of as "non parum curiosum."

Instead, therefore, of proving the divergence of (7), the investigation amounts to a summation of (8).

The reasoning by which John Bernoulli was first led to assert that the sum of the harmonic series was infinite, is equally fallacious.

The investigation is given by James Bernoulli, in a tract *De Seriebus Infinitis*, appended to the *Ars Conjectandi*, p. 250, and he there states that the discovery of the divergence of the series is due to his brother, who obtained it in the following manner.

$$\text{Let}\qquad P = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots + \frac{1}{r+1} + \dots$$

$$= \frac{1}{2} + \frac{2}{6} + \frac{3}{12} \dots + \frac{r}{r(r+1)} + \dots$$

$$Q_1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} \dots + \frac{1}{r(r+1)} + \dots$$

$$Q_2 = \frac{1}{6} + \frac{1}{12} \dots + \frac{1}{r(r+1)} + \dots$$

$$Q_3 = \frac{1}{12} \dots + \frac{1}{r(r+1)} + \dots$$

....

$$\text{then}\qquad P = Q_1 + Q_2 + Q_3 + \dots,$$

but $Q_1 = 1$ $\left(\text{for its } r^{\text{th}} \text{ term} = \frac{1}{r} - \frac{1}{r+1}\right)$; therefore

$$Q_2 = Q_1 - \tfrac{1}{2} = \tfrac{1}{2},\ Q_3 = Q_2 - \tfrac{1}{6} = \tfrac{1}{3},\ Q_4 = \tfrac{1}{4}, \text{ \&c.};$$

$$\text{therefore}\qquad P = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots = 1 + P \dots\dots\dots\dots (10),$$

"unde sequitur seriem $1 + P = P$, totum parti, si finita esset," whence P is infinite.

In this analysis Q_1, Q_2, Q_3 ... differ respectively from $1, \frac{1}{2}, \frac{1}{3}$ ... by quantities which, though indefinitely small, become when added together (the Q's being infinite in number) equal to unity, and reduce (10) to an identity, thus, n being infinite, let

$$P = \frac{1}{2} + \frac{1}{3} \dots + \frac{1}{n},$$

$$\text{then}\quad Q_1 = 1 - \frac{1}{n},\ Q_2 = \frac{1}{2} - \frac{1}{n},\ Q_3 = \frac{1}{3} - \frac{1}{n},\ Q_{n-1} = \frac{1}{n-1} - \frac{1}{n},$$

so that, on substituting in $P = Q_1 + Q_2 \ldots + Q_{n-1}$, we have

$$P = 1 + \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{n} - n \cdot \frac{1}{n}$$

$$= \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{n} = P,$$

an identity; in fact, it is clear *à priori* that no change in the order of adding a series of terms can give any information with regard to their sum.

Montucla, in referring to John Bernoulli's investigation, speaks of it as indirect but very ingenious.* M. Bertrand also in his *Traité de Calcul Différentiel*, p. 231, under the head "Nécessité de la convergence des séries dont on fait usage" quotes it as "un exemple trés remarquable du danger qu'elles présentent." He does not notice the fallacy in the reasoning; but after obtaining the equation $P = P + 1$ he adds "on serait conduit à conclure $1 = 0$, resultat absurde." This seems scarcely the case, $P = P + 1$ gives on solution $P = \frac{1}{1-1}$, whence either $1 = 0$ or $P = \infty$, 'the danger' consists not in using a divergent series, but in treating an infinite quantity as if we knew it to be finite.

In the previously mentioned tract (*Ars Conjectandi*, p. 262) James Bernoulli finds that in the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots$ the odd terms have to the even terms the ratio $\sqrt{2} - 1$ to 1, while it is evident the former are the larger. The explanation is of the same kind as that of the paradox in this note. Bernoulli intimates that he understood the reason of the discrepancy as he proceeds, "cujus ἐναντιοφανείας rationem, etsi ex infiniti natura finito intellectui comprehendi non posse videatur, nos tamen satis perspectam habemus;" though if he had completely explained the difficulty, it is strange that he did not notice the error in his brother's investigation.

* *Histoire des Mathématiques*, t. III., p. 209.

NOTE ON A FUNCTIONAL ELIMINATION.

By *W. H. L. Russell, F.R.S.*

THE following functional elimination, due to Professor Tait, and proposed by him to some members of the British Association at the late meeting, seems so simple and elegant as to be worthy of publication.

It is required to eliminate ϕ between the equations

$$z=\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2,$$

$$0=\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}.$$

Solving the second equation, we have

$$\phi=f_1(x+iy)+f_2(x-iy),$$

whence

$$\frac{d\phi}{dx}=f_1'(x+iy)+f_2'(x-iy),$$

$$\frac{d\phi}{dy}=if_1'(x+iy)-if_2'(x-iy),$$

therefore

$$z=4f_1'(x+iy)f_2'(x-iy),$$

by substituting in the first equation; and

$$\log z=\log\{4f_1'(x+iy)f_2'(x-iy)\}$$

$$=F_1(x+iy)+F_2(x-iy),$$

therefore

$$\frac{d^2(\log z)}{dx^2}+\frac{d^2(\log z)}{dy^2}=0,$$

the resultant of the elimination.

ON THE TRISECTION OF AN ANGLE.

By *R. W. Genese, B.A.*, Scholar of St. John's College, Cambridge.

M. CHASLES in his treatise on *Conic Sections* remarks, that of the many solutions proposed to this problem, perhaps the simplest is the following:

AB (fig. 10) is a circular arc subtending the given angle at the centre O, points P, Q are taken in BA, such that arc$AP = 2$ arcBQ; AP and OQ meet in X.

It is obvious that the locus of X will meet AB at a point of trisection of the arc. Now BQ subtends the same angle at the centre as AP at the circumference; therefore, if AT be the tangent at A, angle $BOQ =$ angle PAT; and by anharmonics, it is readily seen that the locus of X is a conic passing through O and A. By considering various positions of X, it is easy to shew that the directions of the points at infinity on the locus of X are parallel to the bisectors of the angles at T (the intersection of OB and the tangent to arc at A). The locus of X is thus a rectangular hyperbola.

The tangents to the locus at A and O are both perpendicular to OT; therefore AO is a diameter. Also the point T is on the locus.

The above constitutes, as it were, the analysis of the problem. The solution suggested is simple enough to deserve a place in our elementary Conics.

Complete the parallelogram $ATOT'$ (fig. 11), about which describe a rectangular hyperbola. This will meet the circle in four points, one of which, Y, will lie between A and B.

Because OA is a diameter and YT a chord of the rectangular hyperbola, angle $YOT =$ angle $YAT = \frac{1}{2}$ angle YOA. Therefore angle $YOB = \frac{1}{3}$ of given angle.

NOTE ON ANALYTICAL CONICS.

By *R. W. Genese, B.A.*, Scholar of St. John's College, Cambridge.

LET (α, β) be a point on the conic, $\phi(x, y) = 0$; then $\phi\left(\frac{lx + m\alpha}{l+m} . \frac{ly + m\beta}{l+m}\right) = 0$ will be the equation to a similar conic *touching* ϕ at (α, β).

The coefficients of the terms of the second degree in this equation are $\left(\frac{l}{l+m}\right)^2$ of the similar terms in ϕ. Therefore

$$\phi\left(\frac{lx + m\alpha}{l+m} . \frac{ly + m\beta}{l+m}\right) = \left(\frac{l}{l+m}\right)^2 \phi(x, y),$$

which represents a straight line, must be the tangent at (α, β).

1°. Take $l = m$, and we have

$$\phi\left(\frac{x+\alpha}{2} . \frac{y+\beta}{2}\right) = \tfrac{1}{4}\phi(x, y).$$

2°. Multiply by $(l+m)^2$, and then put $m = -l$. We get for the tangent,

$\phi(x, y) =$ *result of substituting* $x - \alpha$, $y - \beta$ *for* x, y *in the terms of second degree in* ϕ.

This is immediately verified by Geometry; for the second member represents parallels to the asymptotes through (α, β) meeting the conic twice at $\alpha\beta$, and where the line at infinity cuts the conic.

FURTHER NOTE ON LAGRANGE'S DEMONSTRATION OF TAYLOR'S THEOREM.

By Professor *Cayley*.

I CANNOT admit that the view of Taylor's theorem taken in my note, p. 22, is in anywise prejudiced by Mr. Wilkinson's remarks, p. 36.

I quite concede that any use of the expression = is faulty which contradicts the axiom "things which are = the same thing are = one another," it being, of course, understood that the three equations (if $A=B$ and $B=C$, then $A=C$) are used in the same sense. Hence, if

$$\frac{1-x^r}{1-x}=1+x+x^2\ldots+x^{r-1}\ \ldots\ldots\ldots\ldots\ldots(1),$$

and if
$$\frac{1}{1-x}=1+x+x^2\ldots+x^{r-1}\ \ldots\ldots\ldots\ldots\ldots(2),$$

then
$$\frac{1-x^r}{1-x}=\frac{1}{1-x}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

an equation which is not true unless $\frac{x^r}{1-x}=0$. But I have never asserted, and do not assert, the equation (2); and in what I did assert

$$\frac{1}{1-x}=1+x+x^2+\ldots.$$

I explain the sense in which the sign = is used; viz. that if by the process of algebraic division we develope $\frac{1}{1-x}$ in ascending powers of x, then that the coefficients of the successive powers of x, however far the expansion is continued, are 1, 1, 1 ..., being a sense which does not imply the numerical equality of two sides of the equation, for every or any numerical value of x. In fact, as regards the application of the axiom to my note, the only three terms which are spoken of are

A, is $f(x+h)$.

B, the expansion hereof by any algebraical process.

C, the expansion as given by Taylor's theorem.

I protest against the assertion, "that no proposition can ever be proved concerning imaginary quantities;" my own

view is, that we have the fundamental conception $i \times i = -1$, presenting no greater difficulty than that of -1 would do to a rational being whose only conception of number was as an answer to the question, "how many?" and that when we have this fundamental conception (i), the proof of any proposition in regard thereto is as intelligible and as easy as that of a proposition in regard to real magnitudes. But the reference to imaginary quantities might be struck out from my Note, and the illustration be made to rest on the case where x or h is a symbol of operation.

ON THE INTEGRALS $\int_0^\infty \sin(x^n)\,dx$ AND $\int_0^\infty \cos(x^n)\,dx$.

By *J. W. L. Glaisher, B.A.*

The integrals $\int_0^\infty \sin(x^n)\,dx$, $\int_0^\infty \cos(x^n)\,dx$ are rendered interesting, not only by the fact of their including the well-known functions $\int_0^\infty \sin x^2\,dx$, $\int_0^\infty \cos x^2\,dx$ as particular cases, but also on account of the discontinuity which separates their finite and infinite values as n passes through certain critical values. They also claim attention as being in a sort of way analogues of the Gamma function in the form

$$\int_0^\infty e^{-x^n}\,dx = \Gamma\left(1+\frac{1}{n}\right).$$

A paper, having their consideration for its subject, was read by the author before the late meeting of the British Association at Edinburgh, in which attention was drawn to some points in the usual methods of evaluating them, and an investigation was given, which at the same time gave their values when finite, and pointed out when they were infinite. A small table of their values and drawings of the curves, the values of the integrals being ordinates and the values of n the abscissæ, were also included. The abstract printed in the Report of the Association contains a short account of the remarks and investigation and the present note consists of the tables and figures, with their description.

In x lies between 1 and ∞, or between -1 and $-\infty$, $\int_0^\infty \sin u^x\,du = \Gamma\left(1+\frac{1}{x}\right)\sin\frac{\pi}{2x}$, but if x lies between 1 and

-1, the value of the integral is infinite; if x lies between 1 and ∞, $\int_0^\infty \cos u^x du = \Gamma\left(1+\frac{1}{x}\right)\cos\frac{\pi}{2x}$, but if between 1 and $-\infty$ its value is infinite. These results are proved in the abstract referred to.

The following table gives the values of the former integral for ten positive and negative arguments, and of the latter integral for ten positive values.

x	$\int_0^\infty \sin(u^x)\,du.$	x	$\int_0^\infty \sin(u^x)\,dx.$
1	*	-1	∞
2	·6266570	-2	$-1·2533140$
3	·4464898	-3	$-·6770591$
4	·3468653	-4	$-·4689467$
5	·2837297	-5	$-·3597669$
6	·2401115	-6	$-·2921516$
7	·2081544	-7	$-·2460562$
8	·1837249	-8	$-·2125806$
9	·1644388	-9	$-·1871508$
10	·1488240	-10	$-·1671704$

x	$\int_0^\infty \cos(u^x)\,du.$	x	$\int_0^\infty \cos(u^x)\,du.$
1	*	6	·8961083
2	·6266570	7	·9119842
3	·7733429	8	·9236474
4	·8374065	9	·9325789
5	·8732304	10	·9396380

These tables were calculated by means of Legendre's well known values of the Gamma function, and the correctness of the calculation of all the quantities involved was verified by the known relation

$$\log\Gamma\left(1+\frac{1}{x}\right)+\log\Gamma\left(1-\frac{1}{x}\right)+\log\sin\frac{\pi}{2x}$$
$$+\log\cos\frac{\pi}{2x}+\log(2x)=\log\pi.$$

Seven-figure logarithms were used, and this identity was generally so nearly satisfied that the two sides did not differ

in any case by more than a unit in the seventh place; it was this agreement which justified the retention of the last figure in the tables, as it is not likely that any are in error by more than a unit in this place. When $x=1$, the values of the integrals are indeterminate. Figs. 12 and 13 represent the curves

$$y=\Gamma\left(1+\frac{1}{x}\right)\sin\frac{\pi}{2x} \text{..................} (1),$$

$$y=\Gamma\left(1+\frac{1}{x}\right)\cos\frac{\pi}{2x} \text{..................} (2),$$

respectively; the parts in each represented by a thick black line being also the finite portions of the curves $y=\int_0^\infty \sin u^x du$, $y=\int_0^\infty \cos u^x dx$ respectively. In fig. 12, all the portion of the former of these curves corresponding to abscissa, lying between A $(x=1)$ and A' $(x=-1)$, are at infinity, while in fig. 13, all the curve of the latter integral corresponding to abscissæ to the left of A $(x=1)$ is at infinity.

It is at once apparent from the tables or from the curves, that in spite of $\Gamma\left(1+\frac{1}{x}\right)$ having a minimum value corresponding to $x=2.16\ldots$, the integrals

$$\int_0^\infty \sin u^x du \text{ and } \int_0^\infty \cos u^x du$$

continually decrease and increase respectively from $x=1$ to $x=\infty$, and that the property $\int_0^\infty \sin u^2 du = \int_0^\infty \cos u^2 du$ is true for $x=2$ alone.

The forms of the curves (1) and (2) between the values $x=\pm 1$, when the integrals have infinite values, are very remarkable.

Considering, first, the portion between 0 and 1 we see that

$$\Gamma\left(1+\frac{1}{x}\right)\sin\frac{\pi}{2x}=\Gamma(1+n)\sin\frac{n\pi}{2},$$

if $\frac{1}{x}=n$, $=(-)^{\frac{n-1}{2}}\,1.2.3\ldots n$, or 0 according as n (supposed integral) is odd or even, while

$$\Gamma\left(1+\frac{1}{x}\right)\cos\frac{\pi}{2x}=(-)^{\frac{1}{2}n}\,1.2.3\ldots n \text{ or } 0,$$

according as n is even or odd; thus, in fig. 12, the branches BC, ED (which are united by a dotted line) meet at about double as far from Ox as the distance to which they are drawn; EF and the other branch connected with it by a dotted line meet at about fifty times the distance to which they are drawn, and the curve continues to form a series of folds which extend to greater and greater distances from Ox, and, at the same time, get flatter and flatter, so that there are an infinite number formed before Oy is reached. A similar explanation applies to fig. 13, the two branches joined by a dotted line meeting at about ten times as far off as the figure shews them.

Considering now the negative portion between $x=0$ and $x=-1$, we see that since $\Gamma(-n)$ (n integral) is infinite, the curve (1) has asymptotes for $x=-\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}\ldots$, and (2) for $x=-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}\ldots$.

If in $\Gamma\left(1+\frac{1}{x}\right)\sin\frac{\pi}{2x}$ we put $x=-\frac{1}{n}$ (n even) the first factor becomes infinite, the second zero, so that we have to investigate the limit to which their products tend; let therefore $\frac{1}{x}=-n+\varepsilon$ and we have

$$-\Gamma(1-n+\varepsilon)\sin\left(\frac{n\pi}{2}-\frac{\pi\varepsilon}{2}\right)=\frac{1}{1-n}\Gamma(2-n+\varepsilon)(-)^{\frac{1}{2}n}\sin\frac{\pi\varepsilon}{2}$$

$$=\frac{(-)^{\frac{1}{2}n+1}}{1.2\ldots n}\Gamma(\varepsilon)\sin\frac{\pi\epsilon}{2}=\frac{(-)^{\frac{1}{2}n+1}}{1.2\ldots n}\Gamma(1+\varepsilon)\frac{\sin\frac{1}{2}\pi\varepsilon}{\varepsilon}=\frac{(-)^{\frac{1}{2}n+1}}{1.2\ldots n}\frac{\pi}{2},$$

when ε is infinitesimal, and similarly if $\frac{1}{x}=-n+\varepsilon$ (n odd) $\Gamma\left(1+\frac{1}{x}\right)\cos\frac{\pi}{2x}=\frac{(-)^{\frac{n-1}{2}}}{1.2\ldots n}\frac{\pi}{2}$; thus, between the asymptotes (represented by dotted lines) in figs. 12 and 13, we have separate branches of the curve, alternately, positive, and negative, whose vertices successively approach nearer to Ox as we move towards Oy, until ultimately they nearly coincide with the origin; the number of asymptotes being infinite. As it is always interesting to compare curves which stand in a relation to one another analogous to that connecting the sine, cosine, and exponential, the curve $y=\int_0^\infty e^{-u^x}du\left\{=\Gamma\left(1+\frac{1}{x}\right)\right.$ for all values of $\left.x\right\}$, has been drawn in fig. 14. After what has been said no explanation of the figure is needed; it may

be remarked, however, that it has asymptotes corresponding to $x = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4} \ldots$

All three curves have been drawn with care, but it has not been thought worth while to make the rather elaborate numerical calculations which would be necessary to trace them with great accuracy; they are all drawn on the same scale, that is to say, OA in each figure is equal to unity.

One point with respect to the values of the integrals should be noticed. From the formulæ

$$\int_0^\infty e^{-ax} \sin bx x^{n-1}\,dx = \frac{\Gamma(n) \sin\left(n \tan^{-1}\dfrac{b}{a}\right)}{(a^2+b^2)^{\frac{1}{2}n}},$$

the value of $\int_0^\infty \sin bx x^{n-1}\,dx$ has frequently been deduced by putting $a = 0$,* but it is to be remarked that although the former result is true for all values of n, owing to the decrease of e^{-ax}, as n increases, more than counterbalancing the increase of x^{n-1}; yet when $a = 0$, there is no diminishing factor, and the last-written integral is obviously infinite if $n > 2$. Similar remarks apply to the corresponding integral involving the cosine.

The results

$$\int_0^\infty {\sin \atop \cos} bx^m . x^n\,dx = \frac{1}{m} b^{-\frac{n+1}{m}} \Gamma\left(\frac{n+1}{m}\right) {\sin \atop \cos}\left(\frac{n+1}{2m}\pi\right),$$

are also not true for all values of m and n. The limits for $\int_0^\infty \sin x . x^n\,dx$ to be finite are that n must be between 0 and -2, and for $\int_0^\infty \sin x . x^n\,dx$, between 0 and -1; while in order that $\int_0^\infty \sin x^m . x^n\,dx$, $\int_0^\infty \cos x^m . x^n\,dx$ may be finite we must have $n+1$ intermediate in value to $-m$ and m, and 0 and m respectively.

We may also notice that when the integrals have been spoken of as being infinite, they are indeterminate as regards sign as well; take as an example

$$\int_0^\infty \cos x . x\,dx = \Big[x \sin x + \cos x\Big]_0^\infty = \infty \sin\infty + \cos\infty - 1,$$

---

* As in DeMorgan's *Diff. and Int. Calc.*, p. 630.

and the first term may be equal to positive or negative infinity, and even for special values of ∞, to zero.

In conclusion it may be remarked that it has been usual to write the integrals in the form

$$\int_0^\infty x^{n-1} \sin x dx, \int_0^\infty x^{n-1} \cos x dx,$$

so that they are not even to be found in de Haan's Tables, under the form in which they appear at the head of this paper.

We have $$\int_0^\infty \sin x^n dx = \frac{1}{n} \int_0^\infty \frac{\sin x}{x^{1-\frac{1}{n}}} dx,$$

and the comparison of the forms of the two curves $y = \sin x^n$, $y = \frac{\sin x}{x^{1-\frac{1}{n}}}$ is interesting; each consists of a series of sinuous folds passing above and below the axis of x; but in the former curve they cut it at equal distances while their heights diminish (fig. 15); in the latter case, the heights of the folds remain constant, and the points of intersection approach each other closely, and ultimately contain an infinitesimal area by the folds approaching infinitely near, not to the axis, but to each other, so as to become perpendicular to the axis (fig. 16).

Trinity College,
Sept. 1, 1871.

ON A PROPERTY OF THE TORSE CIRCUMSCRIBED ABOUT TWO QUADRIC SURFACES.

By Professor *Cayley*.

THE property mentioned by Mr. Townsend in his paper in the August No., "On a Property in the Theory of Confocal Quadrics," may be demonstrated in a form which, it appears to me, better exhibits the foundation and significance of the theorem.

Starting with two given quadric surfaces, the torse circumscribed about these touches each of a singly infinite

series of quadric surfaces, any two of which may be used (instead of the two given surfaces) to determine the torse; in the series are included four conics, one of them in each of the planes of the self-conjugate tetrahedron of the two given surfaces; and if we attend to only two of these conics, the two conics are in fact any two conics whatever, and the torse is the circumscribed torse of the two conics; or, what is the same thing, it is the envelope of the common tangent-planes of the two conics.

Consider now two conics U, U', the planes of which intersect in a line I; and let I meet U in the points L, M, and meet U' in the points L', M': take A the pole of I in regard to the conic U, and A' the pole of I' in regard to the conic U'.

Take T (fig. 17) any point on I, and draw TP touching U in P, and TP' touching U' in P': the points P, P' may be considered as corresponding points on the two conics respectively.

Join AP and produce it to meet the line I in G; the line APG is in fact the polar of T in regard to the conic U (for T being a point on I, the polar of T passes through A; and this polar also passes through P); that is, the points T, G and L, M are harmonics on the line I; whence also in the plane of the conic U' the lines $P'G$, $P'T$ and $P'L$, $P'M$ are harmonic lines through the point P'. It thus appears that in the particular case where the points L, M are the foci of the conic U', the line $P'G$ is the normal at the point P'; and we may say in general that $P'G$ is the quasi-normal at the point P' of the conic U'.

Consider now the torse circumscribed about the conics U, U'; the plane PTP' will represent any plane, and the line PP' any line of this torse: projecting on the plane of U' with the point A as centre of projection, the projection of PP' is the line $P'G$; which, as just seen, is the quasi-normal of the conic U' at the point P'.

The projection of the cuspidal curve is the envelope of line $P'G$, which is the projection of the generating line PP' of the torse—viz. this envelope is the quasi-evolute of the conic U'; which is the theorem in question.

ON THE ANALOGUE AND ITS RECIPROCAL, IN THE THEORY OF QUADRICS, TO A KNOWN PROPERTY AND ITS RECIPROCAL, IN THE THEORY OF CONICS.

By *R. Townsend, M.A., F.R.S.*

THE two reciprocal properties on the theory of conics, that *if A, B, C be the three vertices [or sides] of a triangle, and A', B', C' any three collinear points on [or concurrent lines through] the three opposite sides [or vertices], then every conic which divides [or subtends] harmonically any two of the three segments [or angles] AA', BB', CC', divides [or subtends] harmonically the third also*, are long and well known; and are as often stated as harmonic properties of the tetragram [or tetrastigm] determined by the line [or point] $A'B'C'$ with the three sides [or vertices] of the triangle ABC.

The two analogous properties in the theory of quadrics, that *if A, B, C, D be the four vertices [or faces] of a tetrahedron, and A', B', C', D' any four collinear points on [or planes through] the four opposite faces [or vertices], then every quadric which divides [or subtends] harmonically any three of the four segments [or angles] AA', BB', CC', DD' divides [or subtends] harmonically the fourth also*, (which may of course be stated also as harmonic properties of the hexaplan [or hexastigm] determined by any two planes through [or points on] the line $A'B'C'D'$ with the four faces [or vertices] of the tetrahedron $ABCD$) though implicitly involved in others, also long and well known, have not, so far as I am aware, been given before in a form exhibiting so exactly the analogy between them and the aforesaid properties in the theory of conics to which they correspond.*

* When these analogues first occurred to me in the form above given, I did not at once (nor indeed for some time after) perceive that the properties as above stated were implicitly contained in the well-known reciprocal theorems of Chasles, that *for every pair of tetrahedra, reciprocal polars to each other with respect to any quadric, the four lines of intersection of corresponding faces [or of connection of corresponding vertices], are such that every line intersecting with any three of them intersects with the fourth also*, which were originally given by him as the analogues in the theory of quadrics to the known properties in the theory of conics, that *for every pair of triangles, reciprocal polars to each other with respect to any conic, the three points of intersection of corresponding sides [or lines of connection of corresponding vertices] are collinear [or concurrent]*. As above stated, however, the properties admit probably of more extensive application, and exhibit certainly in a clearer light the analogy in question.

To prove these reciprocal properties directly in the form above given. Denoting by $\alpha_1\beta_1\gamma_1\delta_1$ and $\alpha_2\beta_2\gamma_2\delta_2$ the coordinates, referred to the four faces [or vertices] of the tetrahedron $ABCD$, of any two points P_1 and P_2 on [or planes P_1 and P_2 through] the line $A'B'C'D'$; by $\alpha_A\beta_A\gamma_A\delta_A$, $\alpha_B\beta_B\gamma_B\delta_B$, $\alpha_C\beta_C\gamma_C\delta_C$, $\alpha_D\beta_D\gamma_D\delta_D$, those of the four points [or planes] A, B, C, D, all of which, with the exception of $\alpha_{A'}$, $\beta_{B'}$, $\gamma_{C'}$, $\delta_{D'}$ of course $=0$; by $\alpha_{A'}$, $\beta_{A'}$, $\gamma_{A'}$, $\delta_{A'}$, $\alpha_{B'}$, $\beta_{B'}$, $\gamma_{B'}$, $\delta_{B'}$, $\alpha_{C'}$, $\beta_{C'}$, $\gamma_{C'}$, $\delta_{C'}$, $\alpha_{D'}$, $\beta_{D'}$, $\gamma_{D'}$, $\delta_{D'}$, those of the four points [or planes] A', B', C', D', the four of which $\alpha_{A'}$, $\beta_{B'}$, $\gamma_{C'}$, $\delta_{D'}$, of course $=0$; and by

$$U = a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + 2l\beta\gamma + 2m\gamma\alpha + 2n\alpha\beta$$
$$+ 2p\alpha\delta + 2q\beta\delta + 2r\gamma\delta = 0,$$

the general equation in quadriplanar [or quadripunctal] coordinates $\alpha\beta\gamma\delta$ of any quadric, all referred to the same; then, in the well-known equation of condition that the latter should divide [or subtend] harmonically the segment [or angle] PP', viz.

$$\alpha'\left(\frac{dU}{d\alpha}\right) + \beta'\left(\frac{dU}{d\beta}\right) + \gamma'\left(\frac{dU}{d\gamma}\right) + \delta'\left(\frac{dU}{d\delta}\right) = 0.$$

substituting for the coordinates $\alpha\beta\gamma\delta$, and $\alpha'\beta'\gamma'\delta'$ of P and P' those of A and A', B and B', C and C', D and D' successively, and observing that, while for A, B, C, D, all but α_A, β_B, γ_C, δ_D vanish, and while for A', B', C', D' the four $\alpha_{A'}$, $\beta_{B'}$, $\gamma_{C'}$, $\delta_{D'}$ vanish, the remaining twelve $\beta_{A'}$, &c., for the latter have the following values, viz.,

$$\beta_{A'} = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\alpha_1 - \alpha_2}, \quad \gamma_{A'} = \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\alpha_1 - \alpha_2}, \quad \delta_{A'} = \frac{\alpha_1\delta_2 - \alpha_2\delta_1}{\alpha_1 - \alpha_2},$$

$$\gamma_{B'} = \frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\beta_1 - \beta_2}, \quad \delta_{B'} = \frac{\beta_1\delta_2 - \beta_2\delta_1}{\beta_1 - \beta_2}, \quad \alpha_{B'} = \frac{\beta_1\alpha_2 - \beta_2\alpha_1}{\beta_1 - \beta_2},$$

$$\delta_{C'} = \frac{\gamma_1\delta_2 - \gamma_2\delta_1}{\gamma_1 - \gamma_2}, \quad \alpha_{C'} = \frac{\gamma_1\alpha_2 - \gamma_2\alpha_1}{\gamma_1 - \gamma_2}, \quad \beta_{C'} = \frac{\gamma_1\beta_2 - \gamma_2\beta_1}{\gamma_1 - \gamma_2},$$

$$\alpha_{D'} = \frac{\delta_1\alpha_2 - \delta_2\alpha_1}{\delta_1 - \delta_2}, \quad \beta_{D'} = \frac{\delta_1\beta_2 - \delta_2\beta_1}{\delta_1 - \delta_2}, \quad \gamma_{D'} = \frac{\delta_1\gamma_2 - \delta_2\gamma_1}{\delta_1 - \delta_2},$$

we get at once, between the eight coordinatas $\alpha_1\beta_1\gamma_1\delta_1$ and $\alpha_2\beta_2\gamma_2\delta_2$ of the two points [or planes] P_1 and P_2, and the six coefficients l, m, n, p, q, r in the equation of the quadric, the four following equations of condition for the harmonic section

[or subtense] by the latter of the four segments [or angles] AA', BB', CC', DD', viz.,

$$n(\alpha_1\beta_2-\alpha_2\beta_1)+m(\alpha_1\gamma_2-\alpha_2\gamma_1)+p(\alpha_1\delta_2-\alpha_2\delta_1)=0,$$
$$l(\beta_1\gamma_2-\beta_2\gamma_1)+n(\beta_1\alpha_2-\beta_2\alpha_1)+q(\beta_1\delta_2-\beta_2\delta_1)=0,$$
$$m(\gamma_1\alpha_2-\gamma_2\alpha_1)+l(\gamma_1\beta_2-\gamma_2\beta_1)+r(\gamma_1\delta_2-\gamma_2\delta_1)=0,$$
$$p(\delta_1\alpha_2-\delta_2\alpha_1)+q(\delta_1\beta_2-\delta_2\beta_1)+r(\delta_1\gamma_2-\delta_2\gamma_1)=0,$$

any three of which manifestly involving the fourth, therefore, &c.*

From the form of the preceding equations of condition, being linear in the six coefficients l, m, n, p, q, r, and independent of the four a, b, c, d, in the equation of the quadric, the following are obvious inferences.

1°. *Every quadric $U=0$, for which the six coefficients l, m, n, p, q are all $=0$, divides [or subtends] harmonically every quartet of segments [or angles] AA', BB', CC', DD' determined by the four vertices with four points on the opposite faces [or by the four faces with four planes through the opposite vertices] of the tetrahedron $ABCD$, independently altogether of the colliniarity of the four points [or planes] A', B', C', D'.* As it ought, the tetrahedron being then self-reciprocal with respect to the surface.

2°. *When two different quadrics $U=0$ and $V=0$ divide [or subtend] harmonically the same quartet of segments [or angles] AA', BB', CC', DD', all qnadrics of the system $U+kV=0$ divide [or subtend] them harmonically also.* As they ought, every such system intercepting on [or subtending at] *every* line a system of segments [or angles] in involution, and having consequently a common segment [or angle] of harmonic section.

* It was the identity in form of the above equations of condition with those occurring in Mr. Ferrers' demonstrations of Chasles' theorems (see Salmon's *Geometry of Three Dimensions*, 2nd Edit., Arts. 230 and 231) that first suggested to me the probable identity, in reality if not in form, of the properties established above and there by them.

Assuming Chasles' theorems, which as originally given by him were established on principles purely geometrical, those otherwise proved above independently may be at once inferred from them, on principles also purely geometrical, as follows. For the harmonic section [or subtense] of the four segments [or angles] AA', BB', CC', DD' by any quadric U, it is at once necessary and sufficient that the four points [or planes] A', B', C', D' lie on [or pass through] the four lines of intersection [or of connection] of the four faces [or vertices] of the tetrahedron $ABCD$ with the corresponding faces [or vertices] of its reciprocal with respect to U; but those four lines (by the theorems in question) being such (in either case) that every line which intersects with any three of them intersects with the fourth also, therefore &c.

3°. *Every two quadrics $U=0$ and $V=0$ having the same system of coefficients l, m, n, p, q, r divide [or subtend] harmonically the same quartets of segments [or angles] AA', BB', CC', DD'.* As they ought, the quadric $U-V=0$ then dividing [or subtending] harmonically *all* such quartets indifferently. (See 1° and 2°).

From the properties themselves also, as from their analogues in theory of conics, several inferences may be deduced. The following are a few of the most obvious.

1°. *Every quadric passing through the four points [or touching the four planes] A, B, C, D, and touching three of the four lines AA', BB', CC', DD', touches the fourth also.* For, dividing [or subtending] as it then does, harmonically the three segments [or angles] determining the three it touches, it divides [or subtends] harmonically that determining the fourth, and therefore touches it also.

2°. *Every quadric passing through three of the four points [or touching three of the four planes] A, B, C, D, and touching the corresponding three of the four lines AA', BB', CC', DD', divides [or subtends] the segment [or angle] determining the fourth harmonically.* For, as in the preceding case, dividing [or subtending], as it then does, harmonically the three segments [or angles] determining the three it touches, it divides [or subtends] harmonically that determining the fourth also.

3°. *Every quadric passing through two (or one) of the four points [or touching two (or one) of the four planes] A, B, C, D, touching the corresponding two (or one) of the four lines AA', BB', CC', DD', and dividing [or subtending] harmonically the segment or angle determining either of the remaining two (or the two determining any two of the remaining three), divides [or subtends] harmonically that determining the fourth also.* For, as in the two preceding cases, dividing [or subtending], as it then does, harmonically three of the four segments [or angles], it divides [or subtends] harmonically the fourth also.

4°. *Every ruled quadric passing through two (or one) of the four lines AA', BB', CC', DD', and dividing [or subtending] harmonically the segment or angle determining either of the remaining two (or the two determining any two of the remaining three), divides [or subtends] harmonically that determining the fourth also.* This is manifestly a particular

case of the preceding property 3°, every ruled quadric touching (inflexionally) every line through which it passes.

5°. *The ruled quadric determined by any three of the four lines AA', BB', CC', DD' touches the fourth at its intersection with the line $A'B'C'D'$.* For, by property 2° above, it divides [or subtends] the segment [or angle] determining the fourth harmonically, and, containing (as it manifestly does in either case) the line $A'B'C'D'$, it consequently touches (in either case) the fourth at its intersection with that line.

6°. *The four ruled quadrics determined by the four lines AA', BB', CC', DD', taken in threes, not only intersect in but touch along the line $A'B'C'D'$; which, accordingly, not only intersects but intersects doubly with the four AA', BB', CC', DD'.* For, by the preceding property 5°, the four quadrics touch at each of the four points in which the line $A'B'C'D'$ is met by the four AA', BB', CC', DD', and therefore at every point on that line. (N.B. It will be shewn further on that the four quadrics in question can never coincide).

7°. *Every six points [or planes] of harmonic section P and P', Q and Q', R and R' of any three AA', BB', CC' of the four segments [or angles] AA', BB', CC', DD' determine in general (that is when no three of them are collinear) four conjugate pairs of planes [or points] PQR and $P'Q'R'$, QRP' and $Q'R'P$, RPQ' and $R'P'Q$, PQR' and $P'Q'R$, each dividing [or subtending] harmonically the fourth DD'.* For, the four conjugate pairs of planes [or points] in question constitute each a degenerate quadric dividing [or subtending] harmonically three of the four segments [or angles] AA', BB', CC', DD', and therefore dividing [or subtending] harmonically the fourth also.

8°. *When, of the eight triads of points [or planes] determining the four conjugate pairs of planes [or points] of the preceding property, any triad PQR is collinear, the conjugate triad $P'Q'R'$, if distinct from it, could not be collinear also.* For, from the harmonicism of the three collinear quartets of points [or planes] $AA'PP'$, $BB'QQ'$, $CC'RR'$ on [or through] the three lines AA', BB', CC', the collinearity of the two triads PQR and $P'Q'R'$ in addition to that of $A'B'C'$ would involve that of ABC also, which is, of course, impossible.

9°. *When, of the same, any triad PQR is collinear, the plane [or point] determined by the conjugate triad $P'Q'R'$,*

when distinct from PQR, passes through [or lies on] the fourth line DD′. For, of the two points [or planes] S and S' of harmonic section [or subtense] of the fourth segment [or angle] DD' by the indeterminate plane [or point] PQR and its determinate conjugate $P'Q'R'$ (see preceding property 8°), the former S (which it will be presently shewn can never be collinear with PQR) being then indeterminate, the latter S' is consequently indeterminate also; and therefore, &c.

10°. *When, of eight points [or planes] of harmonic section P and P′, Q and Q′, R and R′, S and S′ of the four segments [or angles] AA′, BB′, CC′, DD′, any four P, Q, R, S (no two of which are conjugate) are complanar [or concurrent], the remaining four P′, Q′, R′, S′ are complanar [or concurrent] also.* For, the plane [or point] $PQRS$ and the plane [or point] $P'Q'R'$ determined by any three P', Q', R' of the remaining four P', Q', R', S' constitute a degenerate quadric dividing [or subtending] harmonically three of the four segments [or angles] AA', BB', CC', DD', and therefore dividing [or subtending] harmonically the fourth also.

11°. *Of eight points [or planes] of harmonic section P and P′, Q and Q′, R and R′, S and S′ of the four segments [or angles] AA′, BB′, CC′, DD′, no quartet P′Q′R′S′, if distinct from its conjugate PQRS, could be collinear* (see properties 6° and 9° preceding). For, in consequence of the harmonicism of the four collinear quartets of points [or planes] $AA'PP'$, $BB'QQ'$, $CC'RR'$, $DD'SS'$ on [or through] the four lines AA', BB', CC', DD', the collinearity of the quartet $P'Q'R'S'$, in addition to that of the quartet $A'B'C'D'$, combined with the complanarity [or concurrence] of the quartet $PQRS$ (which by the preceding property 10° is involved in the collinearity of its conjugate), would involve the complanarity [or concurrence] of the quartet $ABCD$, which is, of course, impossible.

12°. *In every plane [or through every point] E there exists a line L (generally unique) which, connecting with the four pairs of points [or intersecting with the four pairs of planes] A and A′, B and B′, C and C′, D and D′, determines with them four angles [or segments] ALA′, BLB′, CLC′, DLD′ having a common angle [or segment] EE′ of harmonic section, and therefore in involution with each other.* For, if P, Q, R, S be the four points of intersection [or planes of connection] of E with the four lines AA', BB', CC', DD', and P', Q', R', S' the four harmonically conjugate to them with respect to the

four pairs of points [or planes] A and A', B and B', C and C', D and D', the plane [or point] E' determined (see 10°) by the latter four, determines, by its intersection [or connection] with E, a line L manifestly possessing the property in question.

13°. *When, in the preceding property, the plane [or point] E passes through [or lies on] one AA' of the four lines AA', BB', CC', DD', the line L becomes indeterminate, but turns round a point in [or in a plane through] E.* For, while the conjugate P' to its point of intersection [or plane of connection] P with that line AA' becomes indeterminate with P, the triad Q', R', S' conjugate to its three points of intersection [or planes of connection] Q, R, S with the remaining three BB', CC', DD' becomes collinear (see preceding property 9°); consequently, while the line L becomes indeterminate with the conjugate plane [or point] E', it turns round the point of intersection [or lies in the plane of connection] of the line $Q'R'S'$ with the original plane [or point] E.

14°. *The four middle points of the four segments AA', BB', CC', DD' (in the first part of the general property) are complanar.* For, the four harmonically conjugate to them with respect to the four segments are so, and so consequently are they themselves also, by the first part of 10°. This is, of course, the analogue in three dimensions to the long and well-known property in two that, "the three middle points of the three diagonals of a tetragram are collinear."

15°. *The four spheres, of which the four segments AA', BB', CC', DD' (in the first part of the general property) are diameters, have a common radical axis.* For, since (by that part of the general property) every sphere cutting any three of the four segments harmonically cuts the fourth harmonically also, therefore (by a well-known elementary property) every sphere cutting any three of the four spheres of which they are diameters orthogonally cuts the fourth orthogonally also; and therefore, &c. For a different proof on other principles of this property, due originally to Mr. Clifford, and manifestly the analogue in three dimensions to the known property in two that "the three circles on the three diagonals of a tetragram as diameters are coaxal," see *Quarterly Journal of Mathematics*, Vol. VIII., p. 13.

Trinity College, Dublin,
Sept. 29, 1871.

ON THE RECIPROCAL OF A CERTAIN EQUATION OF A CONIC.

By Professor *Cayley*.

THE following formula is useful in various problems relating to conics: the reciprocal equation of the conic

$$\lambda(ax+by+cz)(a'x+b'y+c'z) - \mu(a''x+b''y+c''z)(a'''x+b'''y+c'''z)=0$$

may be written indifferent in either of the forms

$$\left\{\lambda\begin{vmatrix}\xi, & \eta, & \zeta\\ a', & b', & c'\\ a, & b, & c\end{vmatrix}+\mu\begin{vmatrix}\xi, & \eta, & \zeta\\ a'', & b'', & c''\\ a''', & b''', & c'''\end{vmatrix}\right\}^2 + 4\lambda\mu\begin{vmatrix}\xi, & \eta, & \zeta\\ a, & b, & c\\ a''', & b''', & c'''\end{vmatrix}\begin{vmatrix}\xi, & \eta, & \zeta\\ a', & b', & c'\\ a'', & b'', & c''\end{vmatrix}=0,$$

and

$$\left\{\lambda\begin{vmatrix}\xi, & \eta, & \zeta\\ a', & b', & c'\\ a, & b, & c\end{vmatrix}-\mu\begin{vmatrix}\xi, & \eta, & \zeta\\ a'', & b'', & c''\\ a''', & b''', & c'''\end{vmatrix}\right\}^2 + 4\lambda\mu\begin{vmatrix}\xi, & \eta, & \zeta\\ a, & b, & c\\ a'', & b'', & c''\end{vmatrix}\begin{vmatrix}\xi, & \eta, & \zeta\\ a', & b', & c'\\ a''', & b''', & c'''\end{vmatrix}=0.$$

In fact, in the reciprocal equation, seeking for the coefficient of ξ^2, this is

$$\{\lambda(bc'+b'c)-\mu(b''c'''+b'''c'')\}^2-(2\lambda bb'-2\mu b''b''')(2\lambda cc'-2\mu c''c'''),$$

viz. this is

$$\lambda^2(bc'-b'c)^2+\mu^2(b''c'''-b'''c'')^2 + 2\lambda\mu\left\{\begin{matrix}2bb'c''c'''+2b''b'''cc'\\ -(bc'+b'c)(b''c'''+b'''c'')\end{matrix}\right\},$$

or, as it may be written,

$$\{\lambda(bc'-b'c)\pm\mu(b''c'''-b'''c'')\}^2 + 2\lambda\mu\left\{\begin{matrix}2bb'c''c'''+2b''b'''cc'\\ -(bc'+b'c)(b''c'''+b'''c'')\\ \mp(bc'-b'c)(b''c'''-b'''c'')\end{matrix}\right\}.$$

Taking the upper signs, this is

$$\{\lambda(bc'-b'c)+\mu(b''c'''-b'''c'')\}^2 \\ +4\lambda\mu\begin{pmatrix} bb'c''c'''+b''b'''cc' \\ -bc'b''c'''-b'cb'''c'' \end{pmatrix},$$

viz. the term in $\lambda\mu$ is

$$+4\lambda\mu(bc'''-b'''c)(b'c''-b''c').$$

Taking the lower signs, it is

$$\{\lambda(bc'-b'c)-\mu(b''c'''-b'''c'')\}^2 \\ +4\lambda\mu\begin{pmatrix} bb'c''c'''+b''b'''cc' \\ -bc'b'''c''-b'cb''c''' \end{pmatrix},$$

viz. the term in $\lambda\mu$ is

$$+4\lambda\mu(bc''-b''c)(b'c'''-b'''c').$$

And it is thence easy to infer the forms of the other coefficients, and to arrive at the foregoing result.

THE GEOMETRY OF THE RECTANGULAR HYPERBOLA.

By *C. Taylor, M.A.*

THE geometry of the rectangular hyperbola is a subject which is far from having had justice done to it. Not only are its comparatively simple properties commonly presented merely as particular cases of the properties of the hyperbola generally, but its primary angle-property, than which nothing in the elements of conics can be simpler, is worked out by a cumbrous machinery of asymptotes and conjugate hyperbolas, and is even stated in a way which is positively injurious to the young student, since it obscures one of the most striking analogies between the rectangular hyperbola and the circle. In the circle, conjugate radii include a right angle, and hence if θ, ϕ be the inclinations of any two such radii to any third regarded as axis, we have

$$\theta-\phi=\text{a right angle}.$$

In the rectangular hyperbola, if θ, ϕ be the inclinations of any two conjugate diameters to the axis, we have analogously

$$\theta + \phi = \text{a right angle.}$$

Now the occurrence of such analogies between curves widely different suggests the idea of a related *family* of curves, and thus assists the development of the power of generalization. More than this, the form $\theta + \phi = \frac{1}{2}\pi$ has the advantage of simplicity, and does not bring in any extraneous and unnecessarily advanced conception, as does the statement—which is best suited for a *corollary*—that "conjugate diameters are equally inclined to the *asymptotes*." What is there in this form of statement to indicate the relationship between the angle-properties of the circle and the rectangular hyperbola?

Another proposition of which the proof seems to need revising is that

$$QV^2 = CV^2 - CP^2 = PV.P'V,$$

where CV is the abscissa of any point Q on the curve measured along a diameter PCP'. It is usual to prove first of all a particular case of this proposition, viz. the case in which the abscissa is measured along the axis; next, very naturally, to infer the existence of asymptotes; then to prove a further property of asymptotes, and from it to work back to the general relation $QV^2 = CV^2 - CP^2$. But the more natural way of proceeding would be to deduce from the chord-property $QV^2 = CV^2 - CP^2$ the existence and properties of asymptotes, precisely as is commonly done in the case of the principal abscissa and ordinate, and as in analysis we should infer the existence of the asymptotes $x^2 - y^2 = 0$ from the form of the equation $x^2 - y^2 = a^2$, instead of working back from the asymptotes to the equation of the curve. And, be it remarked, that a geometry which reverses the ordinary processes of analysis can be but of slight use, *regarded as an introduction to analysis.* But the elements of geometry are very commonly so regarded; and this may be taken as justifying the principle which I shall attempt to follow out in the treatment of the rectangle hyperbola, viz. that *ceteris paribus*, the order of analysis is to be preferred.* I

---

* See Vol. v., p. 143, of the *Oxford, Cambridge, and Dublin Messenger of Mathematics.*

therefore begin by dividing the theorems to be proved into three classes, according as they have relation to

a chord-properties,
b tangent-properties,
c asymptote-properties.

It is also desirable to keep the analogy between the rectangular hyperbola and the circle steadily in view.

A. CHORD-PROPERTIES.

Let S be the focus, X the foot of the directrix, P any point on the curve, PM a perpendicular on the directrix. Then from the definition $SP = \sqrt{2}.PM$, or

$$SP^2 = 2.PM^2.$$

The centre is determined by taking a point C in SX produced, such that $CX = SX$. We have then the following relations, A being the vertex,

$$\begin{aligned} CS &= 2CX, \\ CS^2 &= 2CA^2, \\ 2CX^2 &= CA^2. \end{aligned}$$

PROP. I. *To prove that $PN^2 + CA^2 = CN^2$, where PN, CN are the principal ordinate and abscissa of any point on the rectangular hyperbola.*

From the definition, $SP^2 = 2NX^2$.

Therefore $PN^2 + (CN - CS)^2 = 2(CN - CX)^2$.

Therefore, adding equals

$$2CN.CS, \qquad 4CN.CX,$$

we have $PN^2 + CN^2 + CS^2 = 2CN^2 + 2CX^2$,

therefore $PN^2 + 2CA^2 = CN^2 + CA^2$,

or $PN^2 + CA^2 = CN^2$.

In the rectangular hyperbola

$$CN^2 - PN^2 = CA^2.$$

In the circle, it follows immediately from the definition that, with a like notation,

$$CN^2 + PN^2 = CA^2.$$

PROP. II. *The locus of the centres of any system of parallel chords is a straight line.*

Let CN, CN' (fig. 18) be the abscissæ of the extremities P, P' of any chord, let O be the centre of the chord, and CM its abscissa. Draw $P'n$ parallel to the axis and terminated by PN.

Then since $$PN^2 = CN^2 - CA^2,$$

and $$P'N'^2 = CN'^2 - CA^2,$$

therefore $$PN^2 - P'N'^2 = CN^2 - CN'^2,$$

or $$PN - P'N' : CN - CN' = CN + CN' : PN + PN',$$

therefore $$Pn : P'n = 2CM : 2OM.$$

Now for parallel chords, $Pn : P'n$ is a constant ratio. Therefore also $CM : OM$ is constant, and the locus of O is a straight line through C.

PROP. III. *Conjugate diameters and chords* are inclined at complementary angles to the axis.*

For since, as in Prop. II.,

$$Pn : P'n = CM : OM,$$

the right-angled triangles PnP', CMO are similar, so that

$$\angle OCM = P'PM$$

$$= \text{complement of } PKN,$$

K being the point in which PP' meets the axis.

In the circle if CP, CD be any two conjugate radii, and if any other radius CA be taken as an axis of reference, then

$$\angle PCA \sim DCA = \text{a right angle.}$$

In the rectangular hyperbola

$$\angle PCA + DCA = \text{a right angle.}$$

PROP. IV. *The angle between any two chords is equal to that between their conjugates.*

For each pair of conjugate chords make complementary angles with the axis, and the difference of any two angles is equal to the difference of their complements.

---

* See *Messenger*, former series, Vol. v., p. 224.

COR. Since supplemental chords are conjugate, it follows at once, that any chord subtends at the extremities of any diameter angles which are either equal or supplementary.

In the circle, angles in the same segment are equal to one another; and the opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles. In other words, any chord subtends at any two points of the curve angles which are either equal or supplementary.

In particular,

Any chord subtends at the extremities of any diameter angles which are either equal or supplementary.

In this special case, the theorem is true of the rectangular hyperbola, as we have seen in the corollary.

PROP. V. *If CV be the abscissa of any point Q on the curve measured along the diameter PCP', then*

$$QV^2 = PV.P'V = CV^2 - CP^2.$$

The chord QQ' (fig. 19), of which V is the centre, subtends supplementary angles at P, P'. [Prop. IV. Cor.

Produce PV to p, so that $pV = PV$.

Then $QPQ'p$ is a parallelogram, and

$$\angle QpQ' = QPQ'$$
$$= \text{supplement of } QP'Q'.$$

Therefore p, Q, P', Q' are concylic, and $QV^2 = P'V.Vp$.

Therefore $QV^2 = P'V.PV = CV^2 - CP^2.$

In this case, the analogy between the rectangular hyperbola and the circle appears sufficiently in the course of the proof above given, for it appears that, by a very simple construction, the case of the one curve is actually reduced to that of the other, so that we have only to quote a property of the latter in order to prove a property of the former.

We might, however, have proceeded thus, without mentioning the circle.

Since QP, QP' are conjugate,

and also QV, PP',

therefore $\angle QPV = P'QV$.

Hence, by similar triangles,

$$QV : PV = P'V : QV,$$

or

$$QV^2 = PV.P'V.$$

ON THE EQUALITY OF CONJUGATE DIAMETERS.

We have seen that

$$QV^2 = CV^2 - CP^2 \text{ .... (i)},$$

where CV is the abscissa of Q referred to a diameter which meets the curve in P.

If Cv be the abscissa of Q measured along the diameter parallel to QV, which diameter does not meet the curve, then since Qv, Cv are equal respectively to CV, QV, we have

$$Cv^2 = Qv^2 - CP^2 \text{ .... (ii).}$$

Now, along the diameter parallel to QV measure Cp, Cp', each equal to CP or CP', then instead of (ii), we may write

$$Qv^2 = Cv^2 + Cp^2.$$

Comparing this with (i), we observe that in each case we have a relation between the ordinate of Q, its abscissa, and a certain constant length measured along the same diameter with the abscissa. But in the one case, the constant length happens to be that of the semi-diameter; hence, we are led to *define* the constant in the other case as the length of the corresponding semi-diameter, and to speak of p, p' as the "extremities" of that diameter, although it does not meet the curve at all.

This appears to be the natural order of proceeding, and it is also commended to us from its being the order of analysis.

In order to make our statement of the second case strictly analogous to that of the first, we should write $-Cp_1^2$ for Cp^2; but this is of course unsuitable in an elementary treatment of the subject. It is usual to say by way of definition, that the extremities of a diameter which does not meet the hyperbola, are the points in which it meets the conjugate hyperbola. This, however, does not throw any new light upon the subject; it does not in the least explain any peculiarites of these non-intersecting diameters: it tends rather to obscure them or explain them away. The properties of the conjugate hyperbola are interesting as corolleries, but as fundamental propositions they are at best useless. They lead to a complication of the diagrams, if not to confusion of thought.

PROP. VI. *If a chord QQ' passes through a fixed point O, then $QO \cdot OQ'$ varies as the square of the parallel radius.*

Let V (fig. 20) be the centre of the chord, q the point in which CO meets the curve, Cv the abscissa of q.

Let CP be the radius parallel to the chord, then since

$$QV^2 + CP^2 = CV^2, \qquad \text{[Prop. v.}$$

and
$$qv^2 + CP^2 = Cv^2,$$

therefore
$$QV^2 + CP^2 : qv^2 + CP^2 = CV^2 : Cv^2$$
$$= OV^2 : qv^2,$$

by similar triangles.

Therefore $QV^2 - OV^2 + CP^2 : CP^2 = OV^2 : qv^2$.

But $OV : qv$ is equal to $CO : Cq$, which is *a constant ratio*, the points O, q being fixed.

Therefore $QV^2 - OV^2 + CP^2 : CP^2$ is a constant ratio.

Therefore $QO.OQ (= QV^2 - OV^2)$ varies as CP^2.

In the circle $QO.OQ$ varies as (radius)^2, *i.e.* it has always the same value.

In the rectangular hyperbola, the rectangle $QO.OQ'$ has the same value for sets of chords whose inclinations to the axis are

$$\theta,\ \tfrac{1}{2}\pi - \theta,\ \pi - \theta,$$

(To be continued).

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

The Annual General Meeting of this Society was held at the Society's Rooms, 22, Albemarle Street, on Thursday Evening, Nov., 9th, at 8 p.m. The number of Members at the commencement of this, the Eighth Session, is 114. Shortly after the President, W. Spottiswoode, Esq., LL.D., F.R.S., had taken the chair, the Meeting proceeded to the election of the Council for the ensuing Session. The result of an examination of the balloting lists by the Scrutators was that the following gentlemen were declared to be elected.—*President:* Dr. Spottiswoode. *Vice-Presidents:* Profs. Cayley, Henrici, H. J. S. Smith, and Mr. Samuel Roberts. *Treasurer:* Dr. Hirst. *Hon. Secs.:* Messrs. M. Jenkins and R. Tucker. *Other Members:* Profs. Clifford and Crofton; Dr. Sylvester; Messrs. T. Cotterill, Merrifield, Stirling, and Walker, and the Hon. J. W. Strutt.

Mr. A. Freeman was proposed for election. The President entered into an explanation of the reasons which had led the Council to propose that an addition should now be made to the number of its Honorary Foreign Members. It being unanimously agreed that it was desirable so to enlarge this class of Members, the Chairman nominated, in the usual way, the following five distinguished Foreign Mathematicians: Dr. Clebsch, M. Hermite, Prof. Cremona, Dr. Hesse, and Prof. Betti. M. Chasles is, at present, the only Honorary Member.

Dr. Sylvester then gave the following abstract of his communication:

"On the partition of an even number into two primes."

In one of his minor papers, Euler has enunciated as a theorem resting entirely on intuition from a comparatively small number of instances that every even number may be decomposed into a sum of two primes. The object of Dr. Sylvester's communication was to obtain some measure of the probable number of ways in which such decomposition can be effected for any given number; if it can be shewn to be probably greater than the square root of the number itself, it will follow, from generally admitted principles of the theory of chances, that the probability of the theorem being universally true, above any assigned limit, if *proved* to be true up to that limit, may be represented by an infinite product of terms which will approach as near as we please to unity the higher the limit is taken.

The mere fact of the theorem, as Euler gave it, being proved up to 100,000,000, or any other number however great, would leave the probability of its being universally true, absolutely zero, just as the fact of the sun having risen 100,000,000 times would not contribute an atom of probability to the supposition that it would continue to rise for all time to come. In the case before us,

on the contrary, the probability of the theorem being universally true, by a sufficiently copious induction, may be made to approach as near as we please to absolute certitude. The author considers that he has established, beyond the reach of reasonable doubt, that the magnitude which represents the mean probable value of the number of modes of effecting the resolution of a very large even number into two prime numbers, is that of the square of the number of primes inferior to the given number divided by the number itself, or which (thanks to the discoveries of Legendre and Tchebicheff) we know to be the same thing, the number of the decompositions in question bears a finite ratio (assignable within limits) to the number to be decomposed, divided by the square of its Napierian logarithm. If we agree provisionally to call preter-primes in respect to n, those numbers which are prime themselves, and also when subtracted from n leave prime remainders, the author shews that the probable number of such preter-primes (*i.e.* the most probable value attainable under our present conditions of knowledge) may be found approximately by multiplying the number of ordinary primes inferior to n by the product of a set of fractions, depending in part on the magnitude, and in part on the constitution of the number n. If n is the double of a prime, the product in question is got by multiplying together all the quantities $\frac{\mu-2}{\mu-1}$ where μ is every odd prime between unity and the square root of n; but if n itself contains any such primes among its factors, then the corresponding factors are to be omitted out of the product. We thus see that if two even numbers of considerable magnitude lie adjacent and tolerably near to each other, one of which is the double of a prime, but the other six times a prime, the number of preter-primes relative to the latter will about twice as many as those relative to the former. For the purpose of greater simplicity of explanation, the formula of approximation has been stated above with less accuracy than it admits of being stated with: instead of the total number of odd primes being multiplied by the product of factors last described, those only should have been taken which are not intermediate between 2 and $\sqrt{(n)}$, and the result so modified should have been stated to be the probable value not of the total number of preter-primes, but only of such of them (by far the larger number) as are not of the excluded class above described, nor subtracted from n, give rise to remainders belonging to such class. The author has found, by actual trial on an extensive scale, that the estimated values of the number of decompositions never differ by more than a moderate, and, in some cases, exceedingly slight, percentage from their actual values determined by the use of Borchardt's tables.

The same methods enable him also to assign a probable value to the number of modes of resolving an odd number into the sum of one prime and the double of another, and in general leads to an approximate representation of the number of solutions in prime numbers of any system of linear equations of which the total number of solutions is limited, and even to resolve approximately such questions as that of determining how many prime numbers there are inferior to a given limit, which are followed by prime numbers differing from them by any assigned interval.

Since the communication made to the Mathematical Society the Secretaries have been informed by Dr. Sylvester that he has verified his results by quite a different method. The exact number of the solutions of the equation $x+y=n$ in prime numbers may be expressed algebraically by means of the method of Generative Functions in terms of the inferior primes to n. The expression will be found to consist of two parts, one a constant multiple of n, the other a function of the roots of unity corresponding to the several inferior primes and their combinations. The former non-periodic part may obviously be regarded as the mean value of the expression, and Dr. Sylvester has found that it is identical with the value obtained by the method of averages previously employed.

In order to prove strictly Euler's theorem, it only remains to shew that the entire expression can never become zero. This, Dr. Sylvester believes he has the means of doing, and, at the same time, of assigning exact limits to the number of solutions in question, but in a matter of so much moment and of such singular interest does not wish to express himself in a more decided manner until he has had the opportunity of subjecting his method to a further rigorous examination.

R. Tucker, M.A., *Hon. Sec.*

ON THE IMPERFECT ELASTICITY OF PERFECTLY ELASTIC RODS.

By *J. Hopkinson, B.A., D.Sc.*, Fellow of Trinity College, Cambridge.

WHEN two perfectly elastic bodies impinge on each other, is it not possible that some portion of the energy of impact should appear, after the impact, in the form of vibrations in the bodies of the nature of sound, and that hence the velocity of separation would be unequal to the velocity of approach, and though the bodies be perfectly elastic, *i.e.* are such that no energy can disappear in them through the agency of friction, may not their modulus of elasticity be less than unity? In the case of elastic rods equal to each other; or, which is the same thing, in the case of a rod impinging on a rigid wall, it has been shewn that this is not the case; the modulus is unity provided there be no internal friction. I propose now to shew that even if the rods impinging be of the same material, yet if their lengths be different, the rods will rebound in a state of vibration, and their centres of gravity will have less relative velocity than Newton's law would lead us to expect of perfectly elastic bodies.

Since we are only concerned with relative motion, let us take the velocities to be inversely proportional to the lengths, so that the centre of gravity of the two rods will be at rest. Take this as origin of abscissa, at the moment of impact let x be abscissa of any point of either rod and $x+\xi$ be abscissa of the same point at the time t during the contact of the rods. Up to the moment when the pressure between the rods vanishes, we may consider the two rods as one of length $2l$ equal to the sum of the lengths, and free at its extremities.

The equation of motion is therefore

$$(1) \qquad \frac{d^2\xi}{dt^2} = a^2 \frac{d^2\xi}{dx^2},$$

when a is velocity of a longitudinal wave in the material.

(2) When $x = \pm l$, $\dfrac{d\xi}{dx} = 0$ for all values of t.

(3) When $t=0$, $\xi=0$, and

$$\frac{d\xi}{dt}=\frac{V}{l+l'} \text{ from } x=-l \text{ to } x=l'$$

and
$$=\frac{V}{-l+l'} \text{ from } x=l' \text{ to } x=l,$$

where $l+l'$, $l-l'$ are the lengths of the rods, and $\frac{V}{l+l'}$ and $\frac{V}{-l+l'}$ their velocities.

The general solution of (1) consistant with (2) is

$$\xi=\Sigma_1^\infty A_n \cos\frac{n\pi}{2l}(x-l)\sin\frac{n\pi}{2l}at.$$

We must now find A_n by condition (3), this gives

$$\Sigma_1^\infty \frac{n\pi a}{2l} A_n \cos\frac{n\pi}{2l}(x-l)=\frac{V}{l+l'} \text{ or } \frac{V}{-l+l'}.$$

Multiply by $\cos\frac{n\pi}{2l}(x-l)$ and integrate from $x=-l$ to $x=l$, we have

$$\frac{n\pi a}{2}A_n=\frac{2l}{n\pi}V\left\{\frac{\sin\frac{n\pi}{2l}(l'-l)}{l+l'}+\frac{\sin\frac{n\pi}{2l}(l'-l)}{-l'+l}\right\},$$

$$A_n=\frac{8l^2.V}{n^2\pi^2(l^2-l'^2)a}\sin\frac{n\pi(l-l')}{2l}.$$

We have then the complete solution of the motion

(4) $\xi=$

$$\frac{8l^2V}{\pi^2a(l^2-l'^2)}\Sigma_1^\infty\frac{1}{n^2}\sin\frac{n\pi(l-l')}{2l}.\cos\frac{n\pi(l-x)}{2l}.\sin\frac{n\pi}{2l}at,$$

thus $\frac{d\xi}{dx}$ will again vanish for all values of x, where $at=2l$, $\frac{2l}{a}$ will be the duration of the impact. It is easily seen that in $\frac{d\xi}{dt}$ the odd terms are the same as when $t=0$, but with sign reversed, but the even terms are the same both in magnitude and sign. Now, if the modulus between the rods were unity, the value of $\frac{d\xi}{dt}$ at any point at the moment of

separation would equal its value where $t = 0$. Referring to equation (4), we see that this can only happen when the even terms vanish, *i.e.* when $l' = 0$, or the rods are equal.

Therefore, if two unequal rods of the same material impinge, their modulus of elasticity will be less than unity, and the rods will rebound in a state of vibration.

The same would probably be true of spheres, and we may infer, that perfect elasticity is in general impossible between unequal bodies, even though their internal friction is nil.

ON THE SQUARES OF TRANSCENDENTS.

By *R. Pendlebury.*

THE methods which have been generally employed by analysts in the evaluation of Definite Integrals are usually reducible, either to a transformation of the variable which is the subject of integration, or to the expansion of the integral in a series, or to the introduction of imaginary quantities into a known integral. An example of a different method is the well-known evaluation of the integral $\int_0^\infty \varepsilon^{-x^2} dx$ bythe expression of its square as a double definite integral, whose value is obtained by a transformation of the variables.* The same method is clearly applicable to any integral we like, though its utility may be diminished by the difficulty of discoveries of suitable transformations to be applied to the multiple integral. I purpose to give some examples in the following paragraphs.

I. Consider the integral $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$. We have clearly

$$\frac{\pi^2}{4} = \int_0^\infty \frac{dx}{1+x^2} \times \int_0^\infty \frac{dy}{1+y^2} = \int_0^\infty \int_0^\infty \frac{dx\,dy}{(1+x^2)(1+y^2)}.$$

Transforming to polar coordinates

$$x = \rho \cos\theta,$$

$$y = \rho \sin\theta,$$

---

* See Todhunter's *Integral Calculus*, Chap. XII.

and $$\frac{\pi^2}{4} = \int_0^\infty \int_0^{\frac{1}{4}\pi} \frac{\rho\, d\rho\, d\theta}{1+\rho^2+\rho^4 \sin^2\theta \cos^2\theta} \ \dots\dots\dots \ \text{(i)}.$$

We can here perform the first integration either with respect to ρ or to θ. Integrating with respect to μ, and making $\rho^2 = r$,

$$\frac{\pi^2}{4} = \frac{1}{2}\int_0^{\frac{1}{4}\pi}\int_0^\infty \frac{dr}{1+r+r^2\sin^2\theta\cos^2\theta}\, d\theta$$

$$= -\frac{1}{4}\int_0^{\frac{1}{4}\pi} \frac{d\theta}{\sin\theta\cos\theta}\cdot\frac{\sin 2\theta}{\cos 2\theta}\log\tan^2\theta$$

$$= -\frac{1}{2}\int_0^{\frac{1}{4}\pi}\frac{d\theta}{\cos 2\theta}\,.\log\tan^2\theta.$$

If we make now $\tan\theta = x$, $d\theta = \frac{dx}{1+x^2}$, $\cos 2\theta = \frac{1-x^2}{1+x^2}$,

$$\frac{\pi^2}{4} = -\int_0^\infty \frac{\log x}{1-x^2}\, dx \ \dots\dots\dots\dots\dots \ \text{(ii)}.$$

Now $$\int_1^\infty \frac{\log x}{1-x^2}\, dx = \int_0^1 \frac{\log x}{1-x^2}\, dx,$$

therefore $$\frac{\pi^2}{8} = -\int_0^1 \frac{\log x}{1-x^2}\, dx \ \dots\dots\dots\dots\dots \ \text{(iii)}.$$

We have in general

$$\int_0^1 x^n \log x = \left[\frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2}\right]_0^1$$

$$= -\frac{1}{(n+1)^2};$$

therefore $$\int_0^1 \frac{\log x}{1-x^2}\, dx = \Sigma_0^\infty \int_0^1 x^n \log x\, dx$$

$$= -\Sigma_0^\infty \frac{1}{(n+1)^2};$$

therefore $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \text{ to } \infty \ \dots\dots\dots \ \text{(iv)},$$

a well-known formula, which is thus proved by a new method.

In the consideration of other integrals, I shall content myself with giving the substitutions and the results without displaying the complete calculation in each case.

II. $$\int_0^a \frac{dx}{1-x^2} = \tfrac{1}{2}\log\frac{1+a}{1-a};$$

therefore $\dfrac{1}{4}\left\{\log\dfrac{1+\alpha}{1-\alpha}\right\}^2$

$$=\left\{\tfrac{1}{2}\int_0^{\frac{1}{4}\pi}\int_0^{\alpha^2\sec^2\theta}+\tfrac{1}{2}\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\int_0^{\alpha^2\operatorname{cosec}^2\theta}\right\}\frac{d\rho\,d\theta}{1-\rho+\rho^2\sin^2\theta\cos^2\theta}.$$

The two integrals are easily combined into one, and we obtain

$$\int_0^{\frac{1}{4}\pi}\log\left\{\frac{\alpha^2\tan^2\theta-1}{\alpha^2-1}\right\}\cdot\frac{d\theta}{\cos 2\theta}=\frac{1}{4}\left\{\log\frac{1+\alpha}{1-\alpha}\right\}^2,$$

and if we put in this $\tan\theta = x$,

$$\int_0^1 \frac{dy}{1-y^2}\log\frac{1-\alpha^2y^2}{1-\alpha^2}=\frac{1}{4}\left(\log\frac{1+\alpha}{1-\alpha}\right)^2 \text{........(v)}.$$

In this formula it is of course supposed that α is less than unity. It may be noticed also, that the subject of integration on the left hand of (v) never becomes infinite.

The truth of formula (v) can be easily verified by differentiating both sides of the equation with respect to α.

III. Consider now the integral

$$\int_0^\alpha \frac{\sin x}{x}\,dx,$$

which has been called the *sine-integral*, and expressed by the notation $Si(\alpha)$.

We have $$\{Si\alpha\}^2=\int_0^\alpha\int_0^\alpha \frac{\sin x.\sin y}{xy}\,dx\,dy$$

$$=\frac{1}{2}\int_0^\alpha\int_0^\alpha\frac{\cos(x-y)-\cos(x+y)}{xy}\,dx\,dy.$$

I transform the integral by the assumption $\xi=\dfrac{x+y}{\sqrt{(2)}}$, $\eta=\dfrac{x-y}{\sqrt{(2)}}$, which are equivalent to the rotation of the axes of coordinates through an angle of 45°. We get then

$$(Si.\alpha)^2=\iint\frac{\cos\{\eta\sqrt{(2)}\}-\cos\{\xi\sqrt{(2)}\}}{\xi^2-\eta^2}\,d\xi\,d\eta,$$

the integral here being extended over the area of a square whose side is a, and of which the axis of η forms a diagonal. Now it is to be remarked, that though the function under the integral sign is never infinite within the area of integration,

yet if we attempt to integrate the separate parts, each becomes infinite along the lines $\xi \pm \eta = 0$, which are two of the sides of the square. To avoid this difficulty I integrate over the area comprised between the lines $\xi - \eta = a\sqrt{(2)}$, $\xi + \eta = a\sqrt{(2)}$, and the lines $\xi \pm \eta = \pm \alpha$. Adding the different definite integrals together, the portions which become infinite when $\alpha = 0$ cancel of themselves, and we may make $\alpha = 0$ in the remainder. The result is (after a slight transformation of limits)

$$(Si\alpha) = \int_0^x \frac{\cos\xi}{\xi} \log\frac{a-\xi}{a}\, d\xi$$

$$+ \int_a^{2a} \frac{\cos\xi}{\xi} \log\frac{\xi - a}{a}\, d\xi + A$$

$$= \int_0^a dy \log\frac{y}{a} \left\{\frac{\cos a\,(1+x)}{1+x} + \frac{\cos a\,(1-x)}{1-x}\right\} + A \ldots \text{(vi)}$$

$$= \int_0^1 dx \log x \left\{\frac{\cos a\,(1+x)}{1+x} + \frac{\cos a\,(1-x)}{1-x}\right\} + A \ldots \text{(vii)}$$

A is a constant, the reasons for the addition of which can be gathered from the integration. Since Sia vanishes with a

$$0 = 2\int_0^1 dx \log x \frac{1}{1-x^2} + A = -\frac{\pi^2}{4} + A$$

by equation (iii). Now $\frac{1}{2}\pi = Si\,\infty$; therefore

$$(Si\,\infty - Si a)(Si\,\infty + Si a)$$

$$= -\int_0^1 dx \log x \left\{\frac{\cos a\,(1+x)}{1+x} + \frac{\cos a\,(1-x)}{1-x}\right\} \ldots \text{(viii)}.$$

We may verify (viii) by differentiation with respect to a,

$$2Si(a)\frac{\sin a}{a} = -2\int_0^1 dx \log x \sin a \cos ax;$$

therefore $$\frac{1}{a} Si(a) = -\int_0^1 \log x \cos ax\, dx,$$

a formula which is obviously true.

In each of the three examples given above the result has been capable of verification by ordinary methods. Indeed it is probable that in any case the method adapted above will be more useful as a means of discovery rather than of proof. By the expression of the square of an integral as a multiple, and the transformation of this multiple integral formulæ may be easy to discover, which would be very

hard to discover though easy to prove by the ordinary methods.

IV. The integral $\int_x^\infty \varepsilon^{-x^2}dx$ or $erf(x)$ can be treated similarly. The result is*

$$\{erfa\}^2 = \varepsilon^{-a^2}\int_1^\infty \frac{\varepsilon^{-x^2x^2}dx}{1+x^2}.$$

V. If we apply the same process to the transcendent

$$\tfrac{1}{2}\pi = \int_0^1 \frac{dx}{\sqrt{(1-x^2)}},$$

we can obtain a formula connected with elliptic integrals. I reserve this, however, for some future opportunity.

Cambridge, *Nov.* 20, 1871.

FURTHER NOTE ON TAYLOR'S THEOREM.

By *M. M. U. Wilkinson, M.A.*

I THINK Prof. Cayley (p. 105) has evaded rather than replied to my objection (p. 36) to his use of the algebraical word =. If an infinite number of factors is as admissible on one side of an equation as an infinite number of terms on the other, of the three general equations

$$\frac{1-xxx...}{1-x} = 1+x+x^2+ \text{..............} \quad (1),$$

$$\frac{1}{1-x} = 1+x+x^2+ \text{..............} \quad (2),$$

$$\frac{1-xxx...}{1-x} = \frac{1}{1-x} \text{.....................} \quad (3),$$

there can be no doubt about (1) being true, and consequently that (3) is a *reductio ad absurdum* of (2). So also, representing by S_r the first $(r+1)$ terms of what is called the expansion in ascending powers of x of $(1-ax)^{-\frac{1}{a}}$, we have, whenever the symbols have real meaning,

$$S_r = (1-ax)^{-\frac{1}{a}}\left\{1 - \frac{(a+1)(2a+1)...(ra+1)}{\underline{|r}}\int_0^x ax^r S_r^{a-1}dx\right\}.$$

In fact, it is not at all difficult to find different functions, such that their expansions when expanded by a definite

---

* *Philosophical Magazine*, December, 1871.

process are the same; and so we have no right to equate all such functions to their expansions, because we should be wrong in equating them to each other. Thus if

$$F(n) = 1 + \frac{y - y^n}{1 - y^{n+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$z = \frac{y}{(1+y)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

we shall have
$$F(n) = \frac{1}{1-}\,\frac{z}{1-}\,\frac{z}{1-\ldots} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

in which continued fraction z occurs $(n-1)$ times. $F(\infty)$ either $= 1 + y$, or $= \dfrac{1+y}{y}$, according as $y^2 <$, or $>$, unity. It can be easily shewn that, in the convergents to $F(n)$, we have the numerator of $F(n) =$ the denominator of $F(n-1)$

$$= \frac{\{1 + \sqrt{(1-4z)}\}^n - \{1 - \sqrt{(1-4z)}\}^n}{2^n \sqrt{(1-4z)}}$$

$$= 1 - (n-2)\, z + \frac{(n-3)(n-4)}{\underline{|2}}\, z^2 + \ldots$$

$$+ (-1)^r \frac{(n-r-1)\ldots(n-2r)}{\underline{|r}}\, z^r + \ldots\ldots\ldots (7),$$

the series stopping at the first zero term. The sum of the series, *ad inf.*, is, when convergent,

$$\frac{\{1 + \sqrt{(1-4z)}\}^n}{2^n \sqrt{(1-4z)}}.$$

But when $4z > 1$, if we assume $4z = \sec^2 \dfrac{\alpha}{2}$, we shall have

$$\text{the numerator of } F(n) = \frac{\sin \dfrac{n\alpha}{2}}{2^{n-2} \cos^{n-2} \dfrac{\alpha}{2} \sin \alpha},$$

and
$$F(n) = \frac{2 \cos \dfrac{\alpha}{2} \sin \dfrac{n\alpha}{2}}{\sin \dfrac{(n+1)\,\alpha}{2}}.$$

I recommend the series (7) to the attention of those who consider it unnecessary, in Euler's proof of the Binomial

Theorem, to restrict the proof to the case when the infinite number of terms which terminate the expansion of $(1+x)^n$ vanish when n is a + integer.

FURTHER NOTE ON TAYLOR'S THEOREM.

By Professor *Cayley*.

ALL that a controversy in general leads to is the exhibition of the precise point of difference between the two parties, and as it does not appear to me that we have yet arrived at this, I make the following further answer:

I did not understand that Mr. Wilkinson objected to what was really my use of the algebraical word =; and I had, therefore, no intention either of evading or of replying to any objection to such use of the word; I considered that he had misapprehended my use of the word, and that by reason of such misapprehension his *reductio ad absurdum* fell to the ground; it was, in my view, as if I having asserted that £1 = 20*s*. (meaning = in value) he had argued that £1 and 1*s*. 6*d*. being each = (in weight) to $\frac{1}{4}$ oz. were therefore = (in weight). The *reductio ad absurdum* against me can only be sustained by Mr. Wilkinson shewing that his use of the word = is the same as my use of it; but it is of course also open for him to contend that my use of the word is a wrong use.

Mr. Wilkinson says there can be no doubt of the truth of the equation

$$\frac{1 - xxx\ldots}{1-x} = 1 + x + x^2 + \ldots \quad \ldots\ldots\ldots\ldots \text{(1)},$$

I cannot attach any definite meaning to this equation; the $(\ldots)$ on the right-hand side refers to the succession of terms $x^3 + x^4 + x^5$ and so on as far as we please; that is, to a succession carried indefinitely onwards of definite terms, and I understand it; but on the left-hand side $xxx\ldots$ means the term obtained as the result of an indefinite number of operations, and this is to me inconceivable: I decline to admit the equation (1).

I reproduce my argument in regard to a particular case as follows: $(x+h)^{\frac{1}{2}}$ is by a definite algebraical process (the ordinary rule for the extraction of the square root) obtained in an ascending series of powers of h; the *same* series is obtained by Taylor's theorem, and in this sense $(x+h)^{\frac{1}{2}}$ is = its expansion as given by Taylor's theorem: any consideration either as to convergency or remainder is irrelevant.

ON CERTAIN THEOREMS IN LOGARITHMIC TRANSCENDENTS.

By *J. W. L. Glaisher, B.A.*

ON p. 45 of this *Essay on the Theory of the various Orders of Logarithmic Transcendents.** Spence has given the formula

$$L_{2n}(1+x) + L_{2n}\left(\frac{1+x}{x}\right)$$
$$= 2L_{2n}(2) + 2L_{2n-2}(2)\frac{L_1^{\,2}(x)}{1.2} + 2L_{2n-4}(2)\frac{L_1^{\,4}(x)}{1.2.3.4}$$
$$+ \ldots \frac{L_1^{\,2n}(x)}{1.2.3\ldots 2n} \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

in which $L_{2n}(1+x)$ is defined to be

$$x - \frac{x^2}{2^{2n}} + \frac{x^3}{3^{2n}} - \ldots .$$

A similar expression is also given for

$$L_{2n-1}(1+x) - L_{2n-1}\left(\frac{1+x}{x}\right),$$

and two analogous formulæ having reference to the functions $x - \frac{x^3}{3^n} + \frac{x^5}{5^n} - \ldots$ are proved on p. 70.

These four results, supplemented by four more, will also be found on pp. 659 and 660 of De Morgan's *Diff. and Int. Calculus.* The method of proof adopted by Spence in regard to the functions $x - \frac{x^2}{2^n} + \frac{x^3}{5^n} - \ldots$ is long and intricate; this objection, however, does not apply to his treatment of the functions $x - \frac{x^3}{3^n} + \frac{x^5}{3^n} - \ldots .$ De Morgan establishes the

---

* Mathematical Essays, by William Spence, edited by Sir J. F. W. Herschel. London, 1819. The first edition was published in 1809. The notation in (1) is not exactly the same as Spence's; he writes the suffix over the letter, thus $\overset{2n}{L}$.

theorems without much difficulty, by considering the well-known expression for the remnant of Taylor's theorem in a definite integral. The following method, however, seems the most direct and easy of application; it also has the advantages of connecting the results when n is odd and even, pointing out the primary series from which the others are derived, and shewing the relation between the results and those obtained by Fourier's theorem.

Taking the first two series as given by De Morgan, we have to shew that

$$(n \text{ even}),\ \left(x - \frac{x^2}{2^n} + \frac{x^3}{3^n} - \dots\right) + \left(x^{-1} - \frac{x^{-2}}{2^n} + \frac{x^{-3}}{3^n} - \dots\right)$$

$$= 2\left\{s_n + s_{n-2}\frac{(\log x)^2}{1.2} \dots + s_2\frac{(\log x)^{n-2}}{1.2.3\dots(n-2)}\right\} + \frac{(\log x)^n}{1.2.3\dots n} \dots (2),$$

$$(n \text{ odd}),\ \left(x - \frac{x^2}{2^n} + \frac{x^3}{3^n} - \dots\right) - \left(x^{-1} - \frac{x^{-2}}{2^n} + \frac{x^{-3}}{3^n} - \dots\right)$$

$$= 2\left\{s_{n-1}\log x + s_{n-3}\frac{(\log x)^3}{1.2.3} \dots + s_2\frac{(\log x)^{n-2}}{1.2.3\dots(n-2)}\right\} + \frac{(\log x)^n}{1.2.3\dots n}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots (3),$$

when $$s_n = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \dots.$$

Consider the series

$$\dots e^{-3\theta} - e^{-2\theta} + e^{-\theta} + e^{\theta} - e^{2\theta} + e^{3\theta} - \dots,$$

whose sum $$= \frac{e^{-\theta}}{1+e^{-\theta}} + \frac{e^{\theta}}{1+e^{\theta}} = 1,$$

so that

$$(e^{\theta} - e^{2\theta} + e^{3\theta} - \dots) + (e^{-\theta} - e^{-2\theta} + e^{-3\theta} - \dots) = 1 \dots\dots\dots (4)$$

Integrate both sides of this equation repeatedly between the limits 0 and θ, and we obtain

$$(e^{\theta} - \tfrac{1}{2}e^{2\theta} + \tfrac{1}{3}e^{3\theta} - \dots) - (e^{-\theta} - \tfrac{1}{2}e^{-2\theta} + \tfrac{1}{3}e^{-3\theta} - \dots) = \theta,$$

$$\left(e^{\theta} - \frac{1}{2^2}e^{2\theta} + \frac{1}{3^2}e^{3\theta} - \dots\right) + \left(e^{-\theta} - \frac{1}{2^2}e^{-2\theta} + \frac{1}{3^2}e^{-3\theta} - \dots\right) = \frac{\theta^2}{1.2} + 2s_2,$$

$$\left(e^{\theta} - \frac{1}{2^3}e^{2\theta} + \frac{1}{3^3}e^{3\theta} - \dots\right) - \left(e^{-\theta} - \frac{1}{2^3}e^{-2\theta} + \frac{1}{3^3}e^{-3\theta} - \dots\right) = \frac{\theta^3}{1.2.3} + 2s_2\theta,$$

and generally

$$\left(e^{\theta}-\frac{1}{2^n}e^{2\theta}+\frac{1}{3^n}e^{3\theta}-\dots\right)+(-)^n\left(e^{-\theta}-\frac{1}{2^n}e^{-2\theta}+\frac{1}{3^n}e^{-3\theta}-\dots\right)$$

$$=2\left\{s_n+s_{n-2}\frac{\theta^2}{1.2}\dots+s_2\frac{\theta^{n-2}}{1.2\dots(n-2)}\right\}+\frac{\theta^n}{1.2\dots n},$$

or $$=2\left\{s_{n-1}\theta+s_{n-3}\frac{\theta^3}{1.2.3}\dots+s_2\frac{\theta^{n-2}}{1.2\dots(n-2)}\right\}+\frac{\theta^n}{1.2\dots n},$$

the first or second series being taken on the right-hand side according as n is even or odd.

By writing $\theta=\log x$ in these results we obtain (2) and (3). Similary by starting with the series

$$(e^{\theta}+e^{2\theta}+e^{3\theta}+\dots)+(e^{-\theta}+e^{-2\theta}+e^{-3\theta}+\dots)=-1,$$

$$(e^{\theta}+e^{3\theta}+e^{5\theta}+\dots)+(e^{-\theta}+e^{-3\theta}+e^{-5\theta}+\dots)=0,$$

$$(e^{\theta}-e^{3\theta}+e^{5\theta}-\dots)-(e^{-\theta}-e^{-3\theta}+e^{-5\theta}-\dots)=0,$$

we establish the other six formulæ given by De Morgan.

A rather more natural course would have been to have taken instead of (4) the corresponding series of cosines, viz.

$$\cos\phi-\cos2\phi+\cos3\phi-\dots=\tfrac{1}{2},$$

or, rather, the arithmetically true formula

$$\sin\phi-\tfrac{1}{2}\sin2\phi+\tfrac{1}{3}\sin3\phi-\dots=\tfrac{1}{2}\phi;$$

integrating this equation between the limits ϕ and 0, we obtain the general formulæ

$$(n\text{ odd}),\ \sin\phi-\frac{1}{2^n}\sin2\phi+\frac{1}{3^n}\sin3\phi-\dots$$

$$=\left\{s_{n-1}\phi-s_{n-3}\frac{\phi^3}{1.2.3}\dots+(-)^{\frac{1}{2}(n-3)}s_2\frac{\phi^{n-2}}{1.2\dots(n-2)}\right\}+(-)^{\frac{1}{2}(n-1)}\tfrac{1}{2}\frac{\phi^n}{1.2\dots n}$$

$$\dots\dots\dots\dots\dots\dots\ (5),$$

$$(n\text{ even}),\ \cos\phi-\frac{1}{2^n}\cos2\phi+\frac{1}{3^n}\cos3\phi-\dots$$

$$=\left\{s_n-s_{n-2}\frac{\phi^2}{1.2}\dots+(-)^{\frac{1}{2}(n-2)}s_2\frac{\phi^{n-2}}{1.2\dots(n-2)}\right\}+(-)^{\frac{1}{2}n}\tfrac{1}{2}\frac{\phi^n}{1.2\dots n}$$

$$\dots\dots\dots\dots\dots\dots\ (6).$$

These formulæ are arithmetically true, and are the proper forms in which (2) and (3) should be presented. The other

true forms corresponding to De Morgan's formula arise by integration from

$$\tfrac{1}{4}\pi = \cos\phi - \tfrac{1}{3}\cos 3\phi + \ldots,\quad \tfrac{1}{4}\pi = \sin\phi + \tfrac{1}{3}\sin 3\phi + \ldots, \text{\&c.}$$

We can thus see the class of theorems to which Spence's belong; for every formula in which a quantity capable of repeated finite integration is equated to a series of sines or cosines, yields on integration results of the same character as those already obtained. Taking, for example, the result

$$\frac{\pi}{2}\frac{e^{a\phi}-e^{-a\phi}}{e^{a\pi}-e^{-a\pi}} = \frac{\sin\phi}{1^2+a^2} - \frac{2\sin 2\phi}{2^2+a^2} + \frac{3\sin 3\phi}{3^2+a^2} - \ldots,$$

we have (n even)

$$\frac{A}{a^{n+1}}\left(e^{a\phi}+e^{-a\phi}\right) - \left(\frac{2A}{a}+\sigma_0\right)\frac{\phi^n}{1.2\ldots n}$$

$$- \left(\frac{2A}{a^3}-\sigma_2\right)\frac{\phi^{n-2}}{1.2\ldots(n-2)} \ldots - \left\{\frac{2A}{a^{n+1}} + (-)^{\frac{1}{2}n}\sigma_n\right\}$$

$$= (-)^{\frac{1}{2}n+1}\left\{\frac{\cos\phi}{1^n(1^2+a^2)} - \frac{\cos 2\phi}{2^n(2^2+a^2)} + \frac{\cos 3\phi}{3^n(3^2+a^2)} - \ldots\right\} \ldots(7),$$

in which A is written for $\dfrac{\pi}{2}\cdot\dfrac{1}{e^{a\pi}-e^{-a\pi}}$, and

$$\sigma_n = \frac{1}{1^n(1^2+a^2)} - \frac{1}{2^n(2^2+a^2)} + \ldots .$$

This result can be verified by making $a=0$, when it coincides with (6).

The corresponding formula for n odd can be at once obtained by differentiating (7). It is unnecessary to write the formula in the same form as Spence's in (2) and (3) as the transformation is effected at once by making $\phi = i\log x$. The equations

$$\frac{\pi}{2a}\frac{e^{a(\pi-x)}+e^{-a(\pi-x)}}{e^{a\pi}-e^{-a\pi}} = \frac{1}{2a^2} + \frac{\cos x}{1^2+a^2} + \frac{\cos 2x}{2^2+a^2} + \ldots,$$

$$\frac{\pi\cos x}{2} = \tfrac{4}{3}\sin 2x + \tfrac{8}{15}\sin 4x \ldots + \frac{1+(-1)^n}{n^2-1}\, n\sin nx + \ldots,$$

and many others of the same kind give rise to similar theorems.

We can also see what are the forms assumed by the formulæ (2) and (3) in the cases when n is respectively odd and even.

From the series

$$\log\left(2\cos\frac{\phi}{2}\right)=\cos\phi-\tfrac{1}{2}\cos 2\phi+\tfrac{1}{3}\cos 3\phi-\dots$$

we find

$$2\log(e^{\frac{1}{2}\theta}+e^{-\frac{1}{2}\theta})=(e^{\theta}-\tfrac{1}{2}e^{2\theta}+\tfrac{1}{3}e^{3\theta}-\dots)+(e^{-\theta}-\tfrac{1}{2}e^{-2\theta}+\tfrac{1}{3}e^{-3\theta}-\dots),$$

whence (n odd)

$$\left(e^{\theta}-\frac{1}{2^n}e^{2\theta}+\frac{1}{3^n}e^{3\theta}-\dots\right)+\left(e^{-\theta}-\frac{1}{2^n}e^{-2\theta}+\frac{1}{3^n}e^{-3\theta}-\dots\right)$$

$$=2\left\{\int_0^{\theta}d\theta\right\}^{n-1}\log(2\cosh\theta)$$

$$+2\left\{s_3\frac{\theta^{n-3}}{1.2\dots(n-3)}+s_5\frac{\theta^{n-5}}{1.2\dots(n-5)}\dots+s_n\right\},$$

and therefore

$$\left(x-\frac{1}{2^n}x^2+\frac{1}{3^n}x^3-\dots\right)+\left(x^{-1}-\frac{1}{2^n}x^{-2}+\frac{1}{3^n}x^{-3}-\dots\right)$$

$$=2\left\{\int_1^x\frac{dx}{x}\right\}^{n-1}\log\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)+2\left\{s_3\frac{(\log x)^{n-3}}{1.2\dots(n-3)}+\dots+s_n\right\},$$

so that when n is odd the formula involves an $(n-1)$-tuple integral.

If we started with the series

$$-\log\left(2\sin\frac{\phi}{2}\right)=\cos\phi+\tfrac{1}{2}\cos 2\phi+\tfrac{1}{3}\cos 3\phi+\dots,$$

we should find on writing θ for ϕi (or we could easily establish independently) that

$$(e^{\theta}+\tfrac{1}{2}e^{2\theta}+\tfrac{1}{3}e^{3\theta}+\dots)+(e^{-\theta}+\tfrac{1}{2}e^{-2\theta}+\tfrac{1}{3}e^{-3\theta}+\dots)$$

$$=-\log(-1)-2\log(e^{\frac{1}{2}\theta}+e^{-\frac{1}{2}\theta}),$$

which involves an imaginary constant.

The series (2) and (3), and others given in this paper, are divergent for all real values of x, so that the formulæ are not arithmetically true. The use which Spence makes of the formulæ (2) and (3) form one among numerous other justifications of the use of Divergent Series in Analysis. He defines $L_n(1+x)$ as $x-\dfrac{x^2}{2^n}+\dfrac{x^3}{3^n}-\dots$; that is to say, he defines $L_n(1+x)$ by a series which is divergent if x is greater than unity. The equations (2) and (3) give him, however, relations connecting $L_n(1+x)$ and $L_n\left(1+\dfrac{1}{x}\right)$, so that the cal-

culation of $L_n(1+x)$ $(x>1)$ is made to depend on the calculation of $L_n(1+x)$ $(x<1)$. Had Spence formally adopted the definition which he has implicitly made use of, viz. that

$$L_n(1+x)=\int_0^x L_{n-1}(1+x)\,\frac{dx}{x},$$

then, replacing $x-\frac{x^2}{2^n}+\frac{x^3}{3^n}-\dots$ by $L_n(1+x)\,(x>1)$, formulæ (2) and (3) would have been unexceptionable from all points of view. As it is they do more than suggest, they conclusively indicate the proper definition in the case when $x>1$.

Spence denotes the series $x-\frac{x^3}{3^n}+\frac{x^5}{5^n}-\dots$ by $C_n(x)$ (he places the n over the C), and the method in which the expressions for $C_n(x)\pm C_n(x^{-1})$ are proved on p. 70 is substantially the same as that adopted in this note; that is to say, the results are obtained by repeated integrations.

It is curious that Abel has devoted a short paper (*Œuvres*, t. II., p. 249) to the investigation of the properties of the series $x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\dots$. At the beginning of the paper he remarks: "Comme la série $x+\frac{x^2}{2^2}+\dots$ n'est convergente que lorsque x ne surpasse pas l'unité, il s'ensuit que la fonction n'a de valeur que pour les x compris entre les limites -1 et $+1$. Pour toute autre valeur de x, la fonction n'existe pas, parce qu'elle est exprimée par une série divergente."

December, 1871.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, Dec. 14th, Dr. Spottiswoode, F.R.S., *President*, in the chair. Mr. A. Freeman, M.A., F.R.A.S., of St. John's College, Cambridge, was elected an Ordinary Member, and the following gentlemen Foreign Members of the Society: Dr. Clebsch, Mr. Hermite, Prof. Cremona, Dr. Hesse, and Prof. Betti.

Dr. Sylvester explained the methods he had employed in his paper "On the theorem that an arithmetical progression which contains more than one, contains an infinite number of prime numbers," of which we give the following abstract furnished by the Author:

This celebrated theorem is one of those which no one would think of doubting, but which are of extreme difficulty to prove. A pretended proof had been given by Legendre a good part of a century ago, and occupies a whole chapter in the *Théorie des Nombres*, but the first real demonstration was accomplished by Lejeune Dirichlet in his great memoir published in the Berlin *Transactions* for the year 1837.

The present communication is limited to the case of Arithmetical Progressions, proceeding according to the common difference 4 or 6. The fundamental theorem employed is an identical equation, on the one side of which are alge-

braical fractions of the form $\frac{x^p}{1-x^{2p}}$, where p represents any combination of the simple powers of any system of primes taken with the positive or negative sign, according as p contains an even or an odd number of factors, and on the other side simple powers of x, whose indices are all the odd numbers not containing any one of the given system of primes as a factor. In the case of progressions with the common difference 4, all the primes of the form $4q+3$ and their primary combinations, figure or indices, on the first side of the equation, and consequently the powers of x on the other side have for their indices combinations of factors of the form $4q+1$. By writing in place of x, the square root of negative unity into x, it is shewn instantaneously that if the number of primes of the form $4q+3$ were finite, a finite series of fractions converging to an infinite value as x approaches to unity would be equal to another such series which would remain finite, which is, of course, absurd. The proof applicable to the case of progressions of the form $4q+1$ is not quite so simple: it depends on shewing that if their number were only i in number it would be possible to have a rational integer function of the degree $2i$ in the logarithm of n greater than a finite multiple of n for a value of n unlimitedly great, which is known to be absurd.

A process precisely similar applies, *mutatis mutandis*, to the case of progressions of the form $6q+1$, $6q+5$; the sole difference being that instead of substituting for x, x multiplied by the square root of negative unity, we must now substitute for it x multiplied successively by the two prime sixth roots of unity and subtract the results from one another. The method here successfully employed in the treatment of these elementary cases appears to differ fundamentally from Dirichlet's method in regard of the circumstance that it deals with an infinitesimal variation in the value of the variable, whereas in Dirichlet's method the infinitesimal variation takes place in the index of the power of the variable.

Observation has shewn it to be in a high degree probable, that the two classes of primes of the form $4q+1$, $4q+3$ respectively, are not only each infinite in number, but that within any assigned limit the number of the one class differs very little from that of the other; that, in other words, the two infinities are in a ratio of equality to one another—but this has never yet been demonstrated, nor has even a demonstration been attempted, and consequently no labour bestowed upon digging around the roots of the question can be considered as superfluous or misapplied; by turning over the soil again and again, and with new implements of research, we may hope to arrive some day at the hidden intuitions necessary to the successful accomplishment of the ulterior objects in view. In fact, in these inquiries connected with the very nature of that true *protoplasm*, which goes by the name of prime numbers, the object of research is not so much the establishment of certain theorems, as the bringing to light the laws of the mind wherein their truth is unconsciously implied. Profs. Cayley and H. J. S. Smith took part in a discussion on this subject. Prof. Clifford next spoke with reference to a paper he is preparing for the Society. Prof. Cayley then drew attention to the question of the determination of the surfaces capable of division into infinitesimal squares by means of their curves of curvature. It was shown by M. Bertrand, that in a triple system of orthotomic isothermal surfaces, each surface possesses the property in question of divisibility into squares by means of its curves of curvature. But in such a triple system, each surface of the system is necessarily a quadric. There is nothing to shew that the property is confined to quadric surfaces; and the question of the determination of the surfaces possessing the property, appears to be one of considerable difficulty, and which has not hitherto been examined. Mr. S. Roberts exhibited a thread model of a homographic transformation of the developable surface which circumscribes a system of confocal quadrics. The surface is generated by planes touching an ellipse at a constant inclination, and its equation is obtained by writing p^2z^2 for r^2 in $\phi(x^2, y^2, r^2) = 0$, representing the plane parallel of the ellipse.

The Rendiconti and Memorie of the Istituto Lombardo and other presents were received for the Society's library. Probable papers for the January meeting (11th instant) are: Prof. Cayley, "On the Surfaces, the Loci of the Vertices of the Cones which satisfy Six Conditions;" Mr. J. W. L. Glaisher (communicated), "On the Constants that occur in certain Summations by Bernoulli's Series;" and Mr. W. Barrett Davis (communicated), "On the Construction of large tables of Divisiors, and of the Factors of the first differences of Prime Powers."

R. Tucker, M.A., *Hon. Sec.*

ON AN IDENTITY IN SPHERICAL TRIGONOMETRY.

By Professor *Cayley*.

In a spherical triangle writing for shortness α, β, γ for the cosines and α', β', γ' for the sines, of the sides; also

$$\Delta^2 = 1 - \alpha^2 - \beta^2 - \gamma^2 + 2\alpha\beta\gamma,$$

we have
$$\cos A = \frac{\alpha - \beta\gamma}{\beta'\gamma'}, \quad \sin A = \frac{\Delta}{\beta'\gamma'},$$

with the like expressions in regard to the other two angles B, C respectively.

Hence

$$\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \&c.$$

$$= \frac{(\alpha-\beta\gamma)(\beta-\gamma\alpha)(\gamma-\alpha\beta) - \Delta^2(\alpha+\beta+\gamma-\beta\gamma-\gamma\alpha-\alpha\beta)}{(1-\alpha^2)(1-\beta^2)(1-\gamma^2)}.$$

The numerator is identically

$$= (1-\alpha)(1-\beta)(1-\gamma)[\Delta^2 - (1+\alpha)(1+\beta)(1+\gamma)],$$

viz. comparing the two expressions, we have

$$(1-\alpha)(1-\beta)(1-\gamma)\Delta^2 - (1-\alpha^2)(1-\beta^2)(1-\gamma^2)$$
$$= (\alpha-\beta\gamma)(\beta-\gamma\alpha)(\gamma-\alpha\beta) + \Delta^2(-\alpha-\beta-\gamma+\beta\gamma+\gamma\alpha+\alpha\beta);$$

or, what is the same thing,

$$(1-\alpha\beta\gamma)\Delta^2 = (1-\alpha^2)(1-\beta^2)(1-\gamma^2) + (\alpha-\beta\gamma)(\beta-\gamma\alpha)(\gamma-\alpha\beta),$$

which is the identity in question and can be immediately verified. We have thus

$$\cos(A+B+C) = \frac{\Delta^2 - (1+\alpha)(1+\beta)(1+\gamma)}{(1+\alpha)(1+\beta)(1+\gamma)},$$

and thence

$$1 + \cos(A+B+C) = \frac{\Delta^2}{(1+\alpha)(1+\beta)(1+\gamma)},$$

$$1 - \cos(A+B+C) = \frac{2(1+\alpha)(1+\beta)(1+\gamma) - \Delta^2}{(1+\alpha)(1+\beta)(1+\gamma)},$$

giving at once the values of $\cos^2\frac{1}{2}(A+B+C)$, $\sin^2\frac{1}{2}(A+B+C)$, $\sin(A+B+C)$, and $\tan^2\frac{1}{2}(A+B+C)$: these are known expressions in regard to the spherical excess.

THE CONVERSE OF PASCAL'S THEOREM.

By *R. W. Genese, B.A.*

I EXTRACT the following note from the *Nouvelles Annales de Mathématiques*, believing that it will prove interesting to the readers of the *Messenger*.

From three points A, B, C on a straight line pairs of straight lines are drawn, shew that any six points of intersection (of which no three are on the same straight line) lie on a conic.

Let $\delta=0$ be the equation to the straight line ABC; and let $\alpha=0$, $\beta=0$, $\gamma=0$ be the equations to three straight lines through A, B, C respectively, so chosen that a harmonic pencil is formed at each point ABC. The equations to the pairs of straight lines will be then

$$\alpha+p\delta=0,\ \alpha-p\delta=0;\ \beta+q\delta=0,\ \beta-q\delta=0;$$
$$\gamma+r\delta=0,\ \gamma-r\delta=0.$$

Consider the curve

$$(\alpha+p\delta)(\beta+q\delta)(\gamma+r\delta)=(\alpha-p\delta)(\beta-q\delta)(\gamma-r\delta).$$

It passes through the intersection of $\alpha+p\delta=0$ with either

$$\alpha-p\delta=0,\ \beta-q\delta=0 \text{ or } \gamma-r\delta=0.$$

That is, it passes through A and two of the points of intersection in question. Similarly it passes through B, C, and four other points of intersection. But on expansion the locus reduces to a line $\delta=0$, and a conic

$$p\beta\gamma+q\gamma\alpha+r\alpha\beta+pqr\delta^2=0.$$

The above six points of intersection not lying on ABC must lie on this conic.

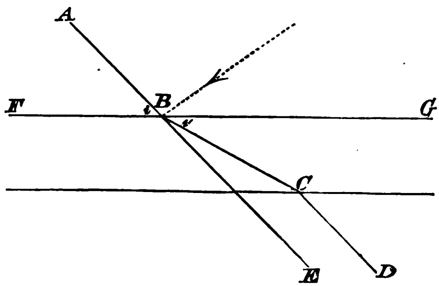
By simply changing the sign of p, q or r, we may obtain other conics on which lie the other sets of six intersections.

ON THE RETARDATION OF A WAVE IN A CRYSTAL.

By *E. J. Routh, M.A.*

THE following method of finding the retardation of a wave in passing through a crystal bounded by parallel planes seems much shorter than that in general use, and which is given in Art. 150 of Airy's *Tract on Light*. The peculiarity of this method is that only a single front is considered, and it is this which simplifies the calculations.

Let $ABCD$ be the front of a wave passing through the crystal, CD being parallel to AB. If there had been no



crystal, the front would have been AB produced. Hence the retardation on emergence is the perpendicular distance between CD and ABE.

Let i, i' be the angles the portions AB, BC of the front make with FBG the face of the crystal. Let T be the thickness of the crystal, then the retardation

$$= \frac{T}{\sin i'} \sin (i - i') = T (\sin i \cot i' - \cos i).$$

If the difference of retardations of the ordinary and extraordinary waves only be required, let i, i'' be corresponding angles for the ordinary wave, then the difference of retardations

$$= T \sin i (\cot i' - \cot i'').$$

If the axis of the crystal be normal to the face, and v, v' be the velocities of the extraordinary wave outside and inside the crystal, we have

$$\left.\begin{array}{l} v'^2 = a^2 \cos^2 i' + c^2 \sin^2 i' \\ \dfrac{v'}{v} = \dfrac{\sin i'}{\sin i} \end{array}\right\}.$$

Therefore eliminating v', we have

$$\cot i' = \frac{\sqrt{(v^2 - c^2 \sin^2 i)}}{a \sin i}.$$

Hence the difference of retardations is

$$\frac{T}{a}\{\sqrt{(v^2 - c^2 \sin^2 i)} - \sqrt{(v^2 - a^2 \sin^2 i)}\},$$

which is the result given in Airy's *Tract.*

In the same way we may find the retardation when the crystal is not cut perpendicular to its axis. We have simply to alter the expression for v'^2.

NOTE ON THE INDICATRIX.

By *R. Pendlebury, B.A.*

In most books of Solid Geometry, the theory of the Indicatrix at any point of a surface, and its relations with the directions of the lines of curvature, and with the magnitude of the radii of curvature form an important part in the general theory of surfaces. It is shewn that this curve is a conic, that the directions of the lines of curvature correspond to the directions of the principal axes of the conic, and that the principal radii of curvature are proportional to the squares of these axes. But I do not know that it has ever been noticed (at least in any elementary "teaching" book) that there exist surfaces for which this theory has cases of exception. At certain points on such surfaces, the indicatrix may be a curve of any form whatever, and the number of lines of curvature passing through these points may be anything, the number of lines of curvature being, in fact, the same as the number of apses in the indicatrix.

An example of such a surface is given by the equation

$$(x^2 + y^2)\,\phi\left(\frac{x}{y}\right) = f(z),$$

where ϕ and f denote any arbitrary functions. For example let the surface

$$(x^2+y^2)=az\phi\left(\frac{y}{x}\right)$$

be considered. This may be supposed to be generated by the revolution of a parabola round its principal axis, the parameter of the parabola increasing or decreasing with the angle through which the plane of the parabola has resolved. Any section of this surface made by a plane perpendicular to the axis (the coordinate-axis of z) will be a curve defined in polar coordinate by the equation

$$\rho^2 = ah\phi\,(\tan\theta),$$

and as the tangent plane at the origin is the plane of xy, this same equation will represent the form of the indicatrix at the vertex. The number of apses of the indicatrix, and of lines of curvature passing through the vertex, will therefore depend on the form of the function ϕ. Indeed, the surface is like an ordinary paraboloid but marked with alternate ridges and furrows radiating from the origin, the summit of each ridge and the bottom of each furrow being a line of curvature.

Let $\phi\,(x, y, z)=0$ be the equation to any surface, if any point on the surface $(\xi\eta\zeta)$ be taken as a new origin the equation to the surface becomes

$$\phi\,(x+\xi,\ y+\eta,\ z+\zeta)=0 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or $$x\frac{d\phi}{d\xi}+y\frac{d\phi}{d\eta}+z\frac{d\phi}{d\zeta}+\frac{1}{\underline{|2}}\left(x\frac{d}{d\xi}+y\frac{d}{d\eta}+z\frac{d}{d\zeta}\right)^2\phi+\&\text{c.}=0\ldots(2).$$

A section made by a plane

$$x\frac{d\phi}{d\xi}+y\frac{d\phi}{d\eta}+z\frac{d\phi}{d\zeta}=h,$$

parallel to and very near the tangent plane at the new origin, will *generally* be similar to the section of the quadric surface

$$x^2\frac{d^2\phi}{d\xi^2}+y^2\frac{d^2\phi}{d\eta^2}+z^2\frac{d^2\phi}{d\zeta^2}+2xy\frac{d^2\phi}{d\xi d\eta}+2yz\frac{d^2\phi}{d\eta d\zeta}+2zx\frac{d^2\phi}{d\xi d\zeta}=\text{const.},$$

and is therefore *generally* a conic. But the theorem fails if either all the second differential coefficients vanish at the point $(\xi\eta\zeta)$, or if the function (1) cannot be expanded at the point $(\xi\eta\zeta)$ in ascending powers of xyz in the form (2).

AN EQUATION IN THE GEOMETRY OF STRAIGHT LINES.

By *Thomas Muir, M.A.*

The equation of the straight line passing through the intersection of the lines $l_1u+m_1v+n_1w=0$, $l_2u+m_2v+n_2w=0$, *and through the intersection of the lines* $l_3u+m_3v+n_3w=0$, $l_4u+m_4v+n_4w=0$ *is*

$$\left\| \begin{array}{ccc} \begin{vmatrix} m_1, & m_2 \\ n_1, & n_2 \end{vmatrix}, & \begin{vmatrix} m_3, & m_4 \\ n_3, & n_4 \end{vmatrix}, & u \\ \begin{vmatrix} n_1, & n_2 \\ l_1, & l_2 \end{vmatrix}, & \begin{vmatrix} n_3, & n_4 \\ l_3, & l_4 \end{vmatrix}, & v \\ \begin{vmatrix} l_1, & l_2 \\ m_1, & m_2 \end{vmatrix}, & \begin{vmatrix} l_3, & l_4 \\ m_3, & m_4 \end{vmatrix}, & w \end{array} \right\| = 0.$$

The equation of *all* lines through the first point of intersection is

$$P(l_1u+m_1v+n_1w) + Q(l_2u+m_2v+n_2w) = 0 \dots (1),$$

and the equation of all lines through the second point of intersection is

$$R(l_3u+m_3v+n_3w) + S(l_4u+m_4v+n_4w) = 0.$$

Now, in order that these equations may denote the same line, viz. the line through both points of intersection, we must have

$$\left.\begin{array}{l} Pl_1 + Ql_2 - Rl_3 - Sl_4 = 0, \\ Pm_1 + Qm_2 - Rm_3 - Sm_4 = 0, \\ Pn_1 + Qn_2 - Rn_3 - Sn_4 = 0, \end{array}\right\},$$

and therefore by elimination of R and S

$$\begin{vmatrix} Pl_1 + Ql_2, & l_3, & l_4 \\ Pm_1 + Qm_2, & m_3, & m_4 \\ Pn_1 + Qn_2, & n_3, & n_4 \end{vmatrix} = 0,$$

whence

$$P \begin{vmatrix} l_1, & l_3, & l_4 \\ m_1, & m_3, & m_4 \\ n_1, & n_3, & n_4 \end{vmatrix} + Q \begin{vmatrix} l_2, & l_3, & l_4 \\ m_2, & m_3, & m_4 \\ n_2, & n_3, & n_4 \end{vmatrix} = 0,$$

which with (1) gives the equation

$$\begin{vmatrix} l_1, & l_3, & l_4 \\ m_1, & m_3, & m_4 \\ n_1, & n_3, & n_4 \end{vmatrix} (l_2u+m_2v+n_2w) - \begin{vmatrix} l_2, & l_3, & l_4 \\ m_2, & m_3, & m_4 \\ n_2, & n_3, & n_4 \end{vmatrix} (l_1u+m_1v+n_1w)=0,$$

or

$$\begin{vmatrix} l_1, & l_2, & o, & o \\ l_1, & l_2, & l_3, & l_4 \\ m_1, & m_2, & m_3, & m_4 \\ n_1, & n_2, & n_3, & n_4 \end{vmatrix} u + \begin{vmatrix} m_1, & m_2, & o, & o \\ l_1, & l_2, & l_3, & l_4 \\ m_1, & m_2, & m_3, & m_4 \\ n_1, & n_2, & n_3, & n_4 \end{vmatrix} v + \begin{vmatrix} n_1, & n_2, & o, & o \\ l_1, & l_2, & l_3, & l_4 \\ m_1, & m_2, & m_3, & m_4 \\ n_1, & n_2, & n_3, & n_4 \end{vmatrix} w=0,$$

whence is derived the form given in the enunciation.

[The equation to the line in question may be obtained more simply as follows: writing, as usual, (m_1n_2) for $\begin{vmatrix} m_1, & n_1 \\ m_2, & n_2 \end{vmatrix}$, &c., the point of intersection of the first two lines is $u : v : w = (m_1n_2) : (n_1l_2) : (l_1m_2)$, and that of the last two lines is $u : v : w = (m_3n_4) : (n_3l_4) : (l_3m_4)$; and Mr. Muir's equation evidently represents the straight line joining these points.—ED.].

Glasgow College,
Jan. 2*nd*, 1872.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

At the meeting of this Society, held Jan. 11th, Dr. Spottiswoode, *President*, and subsequently Prof. Cayley, *Vice-President*, in the chair, Mr. J. W. L. Glaisher, B.A., F.R.A.S., Fellow of Trinity College, Cambridge, was proposed for election, and Major F. Close, R.A., was admitted into the Society. Prof. Cayley gave an account of his paper "On the Surfaces the Loci of the Vertices of the Cones which satisfy Six Conditions." Mr. J. W. L. Glaisher stated and illustrated the principal points in his communication, "On the Constants that occur in Certain Summations by Bernoulli's Series." Mr. W. B. Davis read a paper describing the methods he had used in the construction of Tables of Divisors, and exhibited Tables of Factors of numbers consisting of 9 and 12 figures. A brief discussion ensued on the subject of this communication. Mr. Roberts explained some of the results he submitted to the Society in his paper, "On the Parallel Surfaces of Conicoids and Conics," and illustrated the same by means of a model, and drawings of sections, of one of the surfaces. This surface, in one of its aspects, has a strong resemblance to a cuttle-fish.

R. TUCKER, M.A., *Hon. Sec.*

POINT-RECIPROCATION.

1. *An ellipse (or other conic) reciprocates with respect to a focus into a circle.*

Let S, S', C (fig. 21) be the foci and centre of an ellipse which is to be reciprocated with respect to S. Let fall perpendiculars SY, $S'Y'$ on any tangent, and in the S-perpendicular take a point P such that $SP.SY =$ a constant. Then since $S'Y'.SY =$ another constant, SP varies as $S'Y'$, and the locus of P is therefore similar to the locus of Y', *i.e.* to the auxiliary *circle*.

2. The centre of the auxiliary circle is C. Let O be that of the reciprocal circle. Then by the nature of figures similar and similarly situated, OP is parallel to CY', and

$$\frac{OS}{OP} = \frac{CS'}{CY'} = \frac{ae}{a} = e,$$

where $2a$ and e are the major-axis and the excentricity.

Conversely, if a circle of radius r be reciprocated with respect to a point at a distance d from its centre, *the excentricity of the reciprocal conic is given by the equation*

$$e = \frac{d}{r}.$$

3. *The S-directrix is the polar of O the centre of the reciprocal circle.*

Let X be the foot of the directrix, and $2b$ the minor-axis. Then by a known property

$$CS'.SX = b^2.$$

Let k^2 be the constant of reciprocation, so that $SY.SP = k^2$. Then, since $SY.S'Y' = b^2$,

$$\frac{SO}{CS'} \left(= \frac{SP}{S'Y'} \right)$$

$$= \frac{k^2}{b^2}.$$

Therefore $SX.SO = k^2$, and the directrix is the polar of O.

C. T.

ON THE REDUCTION OF FUNCTIONAL TRANSCENDENTS.

By *J. W. L. Glaisher, B.A., F.R.A.S.*, Fellow of Trinity College, Cambridge.

In Vol. x. of the *Quarterly Journal of Mathematics* I deduced from Boole's General Theorem of Transformation,* the equation

$$\int_{-\infty}^{\infty} \phi f\left(\frac{a_1}{x-\lambda_1}+\frac{a_2}{x-\lambda_2}\dots+\frac{a_n}{x-\lambda_n}\right)dx$$

$$=-\int_{-\infty}^{\infty} dvf(v)\,\Theta\,[\phi]\,\frac{1}{v-\dfrac{a_1}{x-\lambda_1}-\dfrac{a_2}{x-\lambda_2}\dots-\dfrac{a_n}{x-\lambda_n}}\dots(1).$$

Since $\cot x = \dfrac{1}{x}+\dfrac{1}{x-\pi}+\dfrac{1}{x+\pi}+\dots$, we should have

$$\int_{-\infty}^{\infty} \phi f(\cot x)\,dx = -\int_{-\infty}^{\infty} dvf(v)\,\Theta\,[\phi]\,\frac{1}{v-\cot x}\dots(2),$$

but the fact that the terms in the expansion of $(v-\cot x)^{-1}$ in descending powers of x are infinite renders the result of the performance of the operations denoted by Θ uncertain. Two theorems, however, viz. that $\int_{-\infty}^{\infty} f(\cot x)\,dx$ is always infinite if $\int_{-\infty}^{\infty}\dfrac{f(v)}{v^2}\,dv$ be not equal to zero, and that $\int_{-\infty}^{\infty}\dfrac{dx}{x^2}f(\cot x)=\int_{-\infty}^{\infty}f(v)\,dv$, which seemed to rest on better evidence, were given in the paper; and I propose now to verify these results and to investigate by an independent method the general theorem corresponding to (2) after the operation of the symbol Θ; the rest of the paper will be devoted to certain remarks and inferences connected with Boole's Theorem and the theorem (1).

---

* *Phil. Trans.*, 1857, p. 776 et seqq.

If ϕ and f be general functional symbols

$$\int_{-\infty}^{\infty} \phi(x) f(\cot x)\, dx$$
$$= \left\{ \dots \int_{-2\pi}^{-\pi} + \int_{-\pi}^{0} + \int_{0}^{\pi} + \int_{\pi}^{2\pi} + \dots \right\} \phi(x) f(\cot x)\, dx.$$

Now by taking $x = \xi + \pi$ we transform $\int_{\pi}^{2\pi} \phi(x) f(\cot x)\, dx$ into $\int_0^{\pi} \phi(\xi + \pi) f(\cot \xi)\, d\xi$; similarly

$$\int_{2\pi}^{3\pi} \phi(x) f(\cot x)\, dx = \int_0^{\pi} \phi(\xi + 2\pi) f(\cot \xi)\, d\xi,$$

$$\int_{-\pi}^{0} \phi(x) f(\cot x)\, dx = \int_0^{\pi} \phi(\xi - \pi) f(\cot \xi)\, d\xi, \text{ \&c.,}$$

thus we find

$$\int_{-\infty}^{\infty} \phi(x) f(\cot x)\, dx$$
$$= \int_{-\infty}^{\infty} \{\phi(x) + \phi(x - \pi) + \phi(x + \pi) + \dots\} f(\cot x)\, dx$$
$$= \int_{-\infty}^{\infty} \psi(x) f(\cot x)\, dx \quad \dots\dots\dots\dots (3),$$

if $\qquad \psi(x) = \phi(x) + \phi(x - \pi) + \phi(x + \pi) + \dots\,.$

We can also prove that

$$\int_0^{\pi} F(\cot x)\, dx = \int_{-\infty}^{\infty} \frac{F(v)}{1 + v^2}\, dv \quad \dots\dots\dots\dots (4),$$

for the former integral

$$= \int_0^{\frac{1}{2}\pi} F(\cot x)\, dx + \int_{\frac{1}{2}\pi}^{\pi} F(\cot x)\, dx$$
$$= -\int_{x=0}^{x=\frac{1}{2}\pi} \frac{F(\cot x)}{1 + \cot^2 x}\, d(\cot x) - \int_{x=\frac{1}{2}\pi}^{x=\pi} \frac{F(\cot x)}{1 + \cot^2 x}\, d(\cot x)$$
$$= -\int_{\infty}^{0} \frac{F(v)}{1 + v^2}\, dv - \int_0^{-\infty} \frac{F(v)}{1 + v^2}\, dv = \int_{-\infty}^{\infty} \frac{F(v)}{1 + v^2}\, dv.$$

The results (3) and (4) taken together constitute the theorem which takes the place of (2).

Let $\phi(x) = \dfrac{1}{x-h}$ then $\psi(x) = \cot(x-h)$; therefore

$$\int_{-\infty}^{\infty} \frac{dx}{x-h} f(\cot x) = \int_0^{\pi} \cot(x-h) f(\cot x)\, dx,$$

$$= \int_0^{\pi} \frac{1 + \cot h \cot x}{\cot h - \cot x} f(\cot x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1 + v \cot h}{\cot h - v} \frac{f(v)\, dv}{1+v^2}$$

$$= \int_{-\infty}^{\infty} \frac{v + \tan h}{1 - v \tan h} \frac{f(v)\, dv}{1+v^2} \text{ ....(5)},$$

by (4). As a particular case, putting $h = 0$, we find

$$\int_{-\infty}^{\infty} \frac{f(\cot x)}{x} dx = \int_{-\infty}^{\infty} \frac{vf(v)}{1+v^2} dv \text{ ....(6)},$$

whence, for example,

$$\int_0^{\infty} \frac{\sin(a \tan x)}{x} dx = \frac{\pi}{2}(1 - e^{-a}).$$

By taking $\phi(x) = \dfrac{1}{x^2}$, so that $\psi(x) = 1 + \cot^2 x$, or by differentiating (5) with regard to h, and putting $h = 0$, we have

$$\int_{-\infty}^{\infty} \frac{f(\cot x)}{x^2} dx = \int_{-\infty}^{\infty} f(v)\, dv,$$

which is the same equation as that given in Vol. x. of the *Quarterly Journal.*

The general theorem, when $\phi(x) = (x-h)^{-n}$, could easily be found by (3), since $\psi(x) = (-)^n \dfrac{1}{\lfloor n-1} \dfrac{d^{n-1}}{dx^{n-1}}(\cot x)$, but the result is best obtained by differentiating (5) repeatedly with regard to h, whence

$$\int_{-\infty}^{\infty} \frac{dx}{(x-h)^n} f(\cot x) = \frac{1}{\lfloor n-1} \frac{d^{n-1}}{dh^{n-1}} \int_{-\infty}^{\infty} \frac{v + \tan h}{1 - v \tan h} \frac{f(v)\, dv}{1+v^2} \text{ ...(7)}.$$

This equation is really as general as (2); since, in the latter, ϕ must be rational, so as to consist of a series of terms of the forms ax^n and $a(x-h)^{-n}$. If $\phi(x) = ax^n$, $\psi(x)$ is infinite, so that in this case the theorem merely asserts that in general $\int_{-\infty}^{\infty} x^n f(\cot x)\, dx$, ($x^n f(\cot x)$ being an even function

of x), is infinite; and that this is the case is obvious since the integral in question is decomposable into the sum of an infinite series of integrals, each greater than $\int_0^\pi x^n f(\cot x)\,dx$, the value of which is, in general, finite.

For a similar reason $\int_{-\infty}^{\infty} f(\cot x)\,dx$ must be infinite if $\int_0^\pi f(\cot x)\,dx \left\{= \int_{-\infty}^{\infty} \frac{f(v)\,dv}{1+v^2}\right\}$ be not equal to zero. Thus the only form remaining is $a(x-h)^{-n}$ to which (7) applies.

Of course the above reasoning gives no indication of the value of $\int_{-\infty}^{\infty} f(v)\,dx$ when $\int_{-\infty}^{\infty} \frac{f(v)\,dv}{1+v^2}$ is equal to zero. In my previous paper the possibility of a value of the former integral was found to be dependent on the vanishing of $\int_{-\infty}^{\infty} \frac{f(v)}{v^2}\,dv$; the two results are not inconsistent as neither gives really any more information than that $\int_{-\infty}^{\infty} f(\cot x)\,dx$ is, in general, infinite; but if it was necessary to examine a critical case, the formula (2) could not be depended on, for the reason previously stated.

If $\phi(x) = \frac{1}{x^2+a^2}$, we know that

$$\psi(x) = \frac{1}{a}\,\frac{\sinh 2a}{\cosh 2a - \cos 2x},^*$$

and therefore

$$\int_{-\infty}^{\infty} \frac{f(\cot x)}{a^2+x^2}\,dx = \frac{\sinh 2a}{a}\int_0^\pi \frac{f(\cot x)\,dx}{\cosh 2a - \cos 2x} \quad\ldots\ldots(8).$$

The right-hand side, after reduction, of the hyperbolic and circular functions involved, becomes

$$\frac{1}{a}\coth a \int_0^\pi \frac{f(\cot x)\,(1+\cot^2 x)}{\cot^2 x + \coth^2 a}\,dx.$$

Transforming this integral by (4), equation (8) becomes

$$\int_{-\infty}^{\infty} \frac{f(\cot x)}{a^2+x^2}\,dx = \frac{1}{a}\coth a \int_{-\infty}^{\infty} \frac{f(v)\,dv}{v^2+\coth^2 a} \quad\ldots\ldots(9).$$

---

* Sinh x, cosh x, coth x, &c., denote the hyperbolic sine, cosine, cotangent, &c., of x.

By giving special forms to f we have the following interesting integrals:

$$\int_0^\infty \frac{\cos(c\cot x)}{a^2+x^2}\,dx = \frac{\pi}{2a}\,e^{-c\coth a}\,\ldots\ldots\ldots\ldots(10),$$

$$\int_0^\infty \frac{e^{-c^2\cot^2 x}}{a^2+x^2}\,dx = \frac{\sqrt{\pi}}{a}\,e^{c^2\coth^2 a}\,\mathrm{Erf}(c\coth a)\,\ldots\,(11),$$

$$\int_0^\infty \frac{e^{-c^2\cot^2 x}\cos(2b\cot x)}{a^2+x^2}\,dx$$

$$= \frac{\sqrt{\pi}}{2a}\,e^{c^2\coth^2 a}\left\{e^{-2b\coth a}\,\mathrm{Erf}\left(ac-\frac{b}{c}\right)+e^{2b\coth a}\,\mathrm{Erf}\left(ac+\frac{b}{c}\right)\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

and numerous other similar formulæ could be deduced. $\mathrm{Erf}\,x$ is defined as $\int_x^\infty e^{-x^2}dx$; for the formulæ on which (10) and (11) depend, see equations (5) and (14), "On a Class of Definite Integrals," *Phil. Mag.*, October, 1871.

Boole's method gives a result whether all the integrals close up into one or not; thus

$$\left\{\ldots\int_{-\pi+a_1}^{a_2}+\int_{a_1}^{\pi+a_2}+\int_{\pi+a_1}^{2\pi+a_2}+\ldots\right\}\frac{f(\cot x)}{x^2}\,dx=\int_{\cot a_1}^{\cot a_2} f(v)\,dv,$$

but this, and the general case when x^{-2} is replaced by $\phi(x)$ are also included in (3) and (4).

It is to be noticed that whatever difficulties attend the use of (1) and Boole's corresponding theorem when the series $\frac{a_1}{x-\lambda_1}+\frac{a_2}{x-\lambda_2}+\ldots$ extends to infinity, the case when $\phi=1$ may be regarded as free from difficulty. This is apparent from Prof. Cayley's demonstration at the end of the paper in the *Quarterly Journal*. Denoting by $x_1, x_2 \ldots$ the roots of $ax-a_1(x-\lambda_1)^{-1}-a_2(x-\lambda_2)^{-1}-\ldots=v$, we have

$$(x_1-\lambda_1)+(x_2-\lambda_2)+\ldots=\frac{v}{a},$$

and $dx_1+dx_2+\ldots=\frac{dv}{a}$; if therefore $\lambda_1+\lambda_2+\lambda_3+\ldots$ is finite (as in the case of $\cot x$, when it is equal to zero) the proof is quite rigorous; and even when $\lambda_1+\lambda_2+\ldots$ is infinite there seems no reason why its accuracy should be invalidated.

It is clearly necessary, however, both from the investigation and Boole's (*loc. cit.*, p. 777) that when $x=\pm\infty$,

v must $= \pm \infty$; this condition may be regarded as fulfilled in the case of $x - \cot x = v$.

Generally, therefore,

$$\int_{-\infty}^{\infty} f\left(ax - \frac{a_1}{x - \lambda_1} - \frac{a_2}{x - \lambda_2} - \dots\right) dx = \frac{1}{a} \int_{-\infty}^{\infty} f(v)\, dv \dots (13).$$

Now $$\cot x = \frac{1}{x} + \frac{2x}{x^2 - \pi^2} + \frac{2x}{x^2 - 2^2\pi^2} + \dots;$$

therefore $$\frac{1}{\sqrt{x}} \cot \sqrt{x} = \frac{1}{x} + \frac{2}{x - \pi^2} + \frac{2}{x - 2^2\pi^2} + \dots,$$

and $$\int_{-\infty}^{\infty} f\left(x - \frac{1}{\sqrt{x}} \cot \sqrt{x}\right) dx = \int_{-\infty}^{\infty} f(v)\, dv \dots\dots (14).$$

Regarded as a definite integral, the left-hand side of (14) is very remarkable as the quantity subject to the functional sign is of a totally different form over the two portions from ∞ to 0, and from 0 to $-\infty$ of the integral; the quantity $\frac{1}{\sqrt{x}} \cot \sqrt{x}$ being replaced by $-\frac{1}{\sqrt{x}} \coth x$ when x is negative; we may thus write (14) also in the form

$$\int_{-\infty}^{\infty} f\left(x - \frac{1}{\sqrt{x}} \coth \sqrt{x}\right) dx = \int_{-\infty}^{\infty} f(v)\, dv \dots\dots (15),$$

the last result can of course be established independently as

$$\frac{1}{\sqrt{x}} \coth \sqrt{x} = \frac{1}{x} + \frac{2}{x + \pi^2} + \frac{2}{x + 2^2\pi^2} + \dots;$$

(14) is in fact a theorem in the comparison of transcendents and as such would be written

$$\int_0^{\infty} f\left(x - \frac{1}{\sqrt{x}} \cot \sqrt{x}\right) dx + \int_0^{\infty} f\left(-x + \frac{1}{\sqrt{x}} \coth \sqrt{x}\right) dx$$
$$= \int_0^{\infty} f(v)\, dv + \int_0^{\infty} f(-v)\, dv \dots\dots\dots (16).$$

It is in the demonstration of theorems of this class that the peculiar power of Boole's method consists.

If in place of Boole's equation $x - a(x - \lambda)^{-1} - \dots = v$ we were to take the transforming equation

$$ax + \frac{a_1 x}{x + \lambda_1} + \dots + \frac{a_n x}{x + \lambda_n} = v \dots\dots\dots\dots (17),$$

we should find that all the roots were real, and that $\frac{dv}{dx}$ was positive; the only conditions being that a_i and λ_i must have the same sign (viz. so that we might replace any term $ax(x+\lambda)^{-1}$ by $-ax(x-\lambda)^{-1}$). It is also essential that when $x=\pm\infty$, $v=\pm\infty$.

Thus generally, writing $\psi(x)$ for

$$\frac{a_1x}{x+\lambda_1}+\frac{a_2x}{x+\lambda_2}\dots+\frac{a_nx}{x+\lambda_n},$$

$$\int_{-\infty}^{\infty}\phi f\{ax+\psi(x)\}\,dx=\int_{-\infty}^{\infty}\Theta[\phi]\,\frac{dv}{v-ax-\psi(x)}\dots(18),$$

whence $$\int_{-\infty}^{\infty}f\{ax+\psi(x)\}\,dx=\frac{1}{a}\int_{-\infty}^{\infty}f(v)\,dv\dots\dots(19),$$

$$\int_{-\infty}^{\infty}xf\{ax+\psi(x)\}\,dx=\frac{1}{a^2}\int_{-\infty}^{\infty}(v-\Sigma a_i)\,f(v)\,dv\dots(20),$$

$$\int_{-\infty}^{\infty}\frac{dx}{x-h}f\{ax+\psi(x)\}dx=\frac{1}{a}\int_{-\infty}^{\infty}\frac{f(v)\,dv}{v-ah-\psi(h)},\ \&c.\dots(21).$$

If $a=0$, the corresponding results may be proved as in the paper in the *Journal;* we find

$$\int_{-\infty}^{\infty}f\{\psi(x)\}\,dx=(a_1\lambda_1\dots+a_n\lambda_n)\int_{-\infty}^{\infty}\frac{f(v)}{(v-\Sigma a_i)^2}\dots(22),$$

$$\int_{-\infty}^{\infty}xf\{\psi(x)\}\,dx$$

$$=\int_{-\infty}^{\infty}\frac{\Sigma a_i.\,\Sigma a_i\lambda_i^2-(\Sigma a_i\lambda_i)^2-v\Sigma a_i\lambda_i^2}{(v-\Sigma a_i)^3}f(v)\,dv\dots\dots(23),$$

$$\int_{-\infty}^{\infty}\frac{dx}{x-h}f\{\psi(x)\}\,dx$$

$$=\int_{-\infty}^{\infty}\frac{\psi(h)-\Sigma a_i}{v-\psi(h)}\,\frac{f(v)}{v-\Sigma a_i}\,dv,\ \&c.\dots\dots\dots(24).$$

It is preferable to give an independent investigation for the case of $v=ax+a_1x(x_1+\lambda_1)^{-1}+\dots$; but the results in this case are directly deducible from the case when

$$v=ax-a_1(x+\lambda_1)^{-1}-\dots,$$

since $$\frac{a_ix}{x+\lambda_i}=a_i-\frac{a_i\lambda_i}{x+\lambda_i};$$

the conclusions with regard to the signs of a_i and λ_i agree with the results of the direct method.

From the equation

$$\sqrt{x}\cot\sqrt{x} = 1 + \frac{2x}{x-\pi^2} + \frac{2x}{x-2^2\pi^2} + \dots,$$

we have, by (19), since $x + \sqrt{x}\cot\sqrt{x}$ is infinite when x is infinite,

$$\int_{-\infty}^{\infty} f(x - \sqrt{x}\cot\sqrt{x})\,dx = \int_{-\infty}^{\infty} f(x + \sqrt{x}\coth\sqrt{x})\,dx = \int_{-\infty}^{\infty} f(v)\,dv \quad \dots\dots\dots\dots\dots\dots(25).$$

On these integrals the same remarks may be made as on (14) and (15).

Many other integrals of interest insolving a functional sign are deducible from Boole's and other theorems, for instance, in Vol. XI., p. 333, of the *Quarterly Journal*, it is shewn that

$$\Sigma_1^{\infty} \frac{n^2}{n^4+\beta^4} = \frac{\pi}{2\beta\sqrt{2}} \frac{\sinh(\pi\beta\sqrt{2}) - \sin(\pi\beta\sqrt{2})}{\cosh(\pi\beta\sqrt{2}) - \cos(\pi\beta\sqrt{2})},$$

whence we find that

$$\Sigma_1^{\infty} \frac{8\pi^2 n^2}{4\pi^4 n^4 + x} = \frac{1}{x^{\frac{1}{4}}} \frac{\sinh x^{\frac{1}{4}} - \sin x^{\frac{1}{4}}}{\cosh x^{\frac{1}{4}} - \cos x^{\frac{1}{4}}},$$

and therefore

$$\int_{-\infty}^{\infty} f\left(x - \frac{1}{x^{\frac{1}{4}}} \frac{\sinh x^{\frac{1}{4}} - \sin x^{\frac{1}{4}}}{\cosh x^{\frac{1}{4}} - \cosh x^{\frac{1}{4}}}\right) dx = \int_{-\infty}^{\infty} f(v)\,dv,$$

$$\int_{-\infty}^{\infty} f\left(x + x^{\frac{3}{4}} \frac{\sinh x^{\frac{1}{4}} - \sin x^{\frac{1}{4}}}{\cosh x^{\frac{1}{4}} - \cos x^{\frac{1}{4}}}\right) dx = \int_{-\infty}^{\infty} f(v)\,dv, \text{ \&c.}$$

It is easy to see that if $E = 0$ denote any equation with all its roots real, then $x - \frac{1}{E}\frac{dE}{dx} = v$ has all its roots real too; thence it follows that $E(x-v) - E' = 0$ also has all its roots real; we can therefore use

$$x - \frac{1}{E(x-v) - E'} \frac{d}{dx}\{E(x-v)\,E'\} = v_1 \dots\dots(26),$$

as a transforming equation, since $\frac{dv_1}{dx}$ is positive for the same

reason that $\frac{dv}{dx}$ is positive in the previous equation, and the roots of (26) are also all real. Proceeding in this way we could obtain an infinite number of transforming equations; as an example, take $E = \cot x$, then since the equation $x - \cot x = 0$ has all its roots real, $x \sin x - \cos x = 0$ has all its roots real too; we can therefore take

$$x - \frac{2 \sin x + x \cos x}{x \sin x - \cos x} = v$$

as transforming equation; whence we find that

$$\int_{-\infty}^{\infty} f\left(\frac{x^2 - 2x \cot x - 2}{x - \cot x}\right) dx = \int_{-\infty}^{\infty} f(x)\, dx \ \ldots(27).$$

The quantity under the functional sign in this equation not being of the form $\phi(x - \cot x)$, (27) is not deducible from Boole's equation (2), *Phil. Trans.*, loc. cit., p. 789. By the aid of this equation, however, we should find

$$\int_{-\infty}^{\infty} f\left(\frac{x^2 - 2x \cot x + \cot^2 x - a^2}{x - \cot x}\right) dx = \int_{-\infty}^{\infty} f\left(v - \frac{a^2}{v}\right) dv$$

$$= \int_{-\infty}^{\infty} f(v)\, dv,$$

since the quantity under the first functional sign is

$$x - \cot x - a^2 (x - \cot x)^{-1}.$$

It is worthy of notice that it is immaterial whether the equations have positive or negative signs (*i.e.* whether we take $E = 0$ or $-E = 0$), as the resulting equations are unaltered, for $\frac{d}{dx} \log E = \frac{d}{dx} \log(-E)$.

It might at first sight appear that a series of important deductions might be drawn from (13), and similar equations by replacing a_i by $F(u)\, du$, and λ_i by $\chi(u)$, so that

$$x - \frac{a_1}{x - \lambda_1} - \ldots \frac{a_n}{x - \lambda_n} = x - \int_{\alpha}^{\beta} \frac{F(u)\, du}{x - \chi(u)};$$

an examination, however, of the manner in which (13) is established shews that this transformation is not justifiable; it is essential for the truth of the theorem that $x - a(x - \lambda_1)^{-1} - \ldots$ should become infinite when $x = \lambda_1, \lambda_2, \ldots$, but

$$\frac{hF(\alpha)}{x - \chi(\alpha)} + \frac{hF(\alpha + h)}{x - \chi(\alpha + h)} + \ldots \frac{hF(\beta)}{x - \chi(\beta)},$$

in the limit when it $=\int_\alpha^\beta \frac{F(u)\,du}{x-\chi(u)}$ does not vanish for $x=\chi(\alpha),\ \chi(\alpha+h)\ldots$.

It also might appear that the same reasoning that established Boole's theorem in regard to $x-\cot x$ would apply also when $x+\cot x$ was the quantity involved, since $\frac{d}{dx}(x+\cot x)=-\cot^2 x$, and is of invariable sign; the equation $x+\cot x=v$ has also an infinite number of real roots; but it can be shewn by writing $\alpha+\beta i$ for x, and reducing so that the equation takes the form $A+Bi=0$, that the equations $A=0$, $B=0$ can be satisfied by real values of α and β;* the equation $x+\cot x=v$ has in fact two imaginary roots as the discussion of the equation

$$x+(x-\pi)^{-1}+(x+\pi)^{-1}+\ldots=v$$

would lead us to suppose. We have no theorem, therefore, involving $x+\cot x$.

With regard to the equation $x-\cot x=v$, by putting $x=\alpha+\beta i$ as before, we can shew that all its roots are real; an examination of the roots of the equation

$$x-(x-\pi)^{-1}-(x+\pi)^{-1}-\ldots=v$$

shews this equally well, though it involves some curious considerations in reference to infinity.

It may be remarked in conclusion that many points in regard to the general formulæ which have been considered are by no means free from difficulty, as *e.g.* by differentiating (13) with regard to λ_i we should find

$$\int_{-\infty}^{\infty} \frac{dx}{(x-\lambda_i)^2} f'\left(ax-\frac{a_1}{x-\lambda_1}\ldots-\frac{a_n}{x-\lambda_n}\right)dx=0.$$

Some of these I hope to explain on a future occasion.

Jan. 1, 1872.

---

* We can easily see that the equation $x+\cot x=0$ has two roots of the form βi, for putting $x=\beta i$, we have to determine β from the equation $\beta=\frac{e^\beta+e^{-\beta}}{e^\beta-e^{-\beta}}$, viz.

$$\beta\left(\beta+\frac{\beta^3}{6}+\ldots\right)=1+\frac{\beta^2}{2}+\frac{\beta^4}{24}+\ldots,$$

viz. $1-\frac{\beta^2}{2}-\frac{\beta^4}{8}-\ldots=0$, which can evidently be satisfied by a positive value of β^2.

CHANCE.

By *W. Allen Whitworth, M.A.*

If a person who has not had a mathematical education be enquiring concerning the chance of a stated result occurring, he will not ask, How many times will the trial fail for once that it succeeds? a question which would be simply answered by stating the mathematical odds against the event; but he is more likely to ask the question, How many trials must (on an average) be made before the event occurs? and he will expect an answer in the form, On an average you may expect to succeed at the x^{th} trial.

If the repetitions be made always under the same circumstances, so that the chance of succeeding is the same for all the trials, the distinction between the two questions which I have stated is unimportant; for, in this case, if the trial fail $n-1$ times for once that it succeeds, the experimenter will *on an average* have to make n trials in order to gain the desired event. The one question is as easily answered as the other.

But if the chance of success vary at successive trials, it is not always easy to deduce the answer to the second question from the answer to the first.

Throughout the present paper we shall use the letter A to denote the number required in answer to the second form of question; that is, on an average the experiment will succeed at the A^{th} trial.

If $f_1 f_2 f_3 \ldots$ be the respective chances of failing, and $p_1 p_2 p_3 \ldots$ the chances of succeeding at the 1st, 2nd, 3rd... trials, each trial implying that all previous ones have failed, then we may evidently write

$$A = p_1 + 2f_1 p_2 + 3f_1 f_2 p_3 + 4f_1 f_2 f_3 p_4 + \ldots .$$

This formula we shall quote as Prop. I.

By substituting $p_1 = 1 - f_1$, $p_2 = 1 - f_2$, &c., we get

$$A = 1 + f_1 + f_1 f_2 + f_1 f_2 f_3 \ldots .$$

This formula we shall quote as Prop. II.

In many cases there will be no limit to the number of possible trials; the series in each formula will then have to be summed to an infinite number of terms. The series are obviously convergent, f and p being proper fractions. In other cases, success will be certain within a given number

n of trials, in which cases $f_n = 0$, and all terms after the n^{th} will then vanish. But in some cases the number of trials will be limited (n suppose) without success being certain. In these cases we shall have n terms in the series of Prop. I, and the result in Prop. II. will have to be written

$$A = 1 + f_1 + f_1 f_2 + f_1 f_2 f_3 + \ldots + f_1 f_2 \cdots f_{n-1} - n f_1 f_2 \cdots f_n,$$

or more conveniently

$$A = 1 + f_1 + f_1 f_2 + f_1 f_2 f_3 + \ldots + f_1 f_2 \cdots f_n - (n+1) f_1 f_2 \cdots f_n.$$

This result we shall quote as Prop. III.

If $f_1 = f_2 = f_3 = \ldots = f$, we get, in Prop. II.,

$$A = 1 + f + f^2 + f^3 + \ldots.$$

$$= \frac{1}{1-f},$$

or if $$f = 1 - \frac{1}{n}, \quad A = n.$$

That is, if one trial out of n succeeds, it will on average require n trials to produce success. This result, which was stated without proof in the second paragraph, is almost axiomatic. We shall, however, quote it as Prop. IV.

Let C_r denote the chance of succeeding at the r^{th} trial and no sooner, and let D_r denote the chance of succeeding at the r^{th} trial or sooner, and let n be the greatest possible number of trials, then

$$A = C_1 + 2C_2 + 3C_3 + \ldots + nC_n,$$

also
$$D_1 = C_1,$$
$$D_2 = C_1 + C_2,$$
$$D_3 = C_1 + C_2 + C_3,$$

and so on; therefore

$$D_1 + D_2 + D_3 + \ldots + D_n = nC_1 + \overline{n-1}\,C_1 + \ldots + C_n$$
$$= \overline{n+1}\,D_n - A.$$

Hence $$A = (n+1) D_n - \Sigma D.$$

This result we shall quote as Prop. V.

Except in the rare cases, in which the number of trials is limited without success being certain, we shall have $D_n = 1$.

Hence $$A = n + 1 - \Sigma D;$$

we will call this Prop. VI.

EXAMPLES AND ILLUSTRATIONS.

1. A man undertakes to toss a coin so that it may fall '*head*.' On an average he will succeed at the *second* trial (by Prop. IV).

2. A man throws a pair of dice. On an average he will succeed in throwing doublets at the sixth trial (by Prop. IV).

3. A bag contains balls, half of them black and the rest white. He goes on drawing a ball and replacing it until he has drawn two of the same colour consecutively. On an average he will have to draw *three* balls.

For with notation of Prop. II.

$$f_1 = 1,\ f_2 = \tfrac{1}{2},\ f_3 = \tfrac{1}{2},\ f_4 = \tfrac{1}{2},\ \&c.;$$

therefore $$A = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots = 3.$$

Or more simply: after the first ball is drawn the case comes under Prop. IV., which shews that two *more* balls must be drawn.

4. A bag contains n balls, one of which is white. A man draws a ball and replaces until he draws the white one. On an average he will draw n times. (by Prop. IV.)

5. The same case as the last, but the balls not replaced. On an average he will draw $\frac{1}{2}(n+1)$ times. (By Prop. I. or Prop. II).

6. Out of a pack of n cards a card is drawn at random and replaced. The operation is repeated until a card has been drawn *twice*. The average number of drawings will be with the notation of *Choice and Chance*, p. 185,

$$\frac{\underline{|n}}{n^n} e_n^{\ n}.$$

7. A bag contains n black balls and n white ones. A ball is drawn and not replaced, and the operation is repeated until two balls of the same colour are consecutively drawn. (By Prop. III).

$$A + 1 = 3\left\{1 + \frac{1}{2}\,\frac{n}{2n-1} + \frac{1}{4}\,\frac{n(n-1)}{(2n-1)(2n-3)}\right.$$

$$\left. + \frac{1}{8}\,\frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)} + \ldots + \frac{\underline{|n}\,\underline{|n}}{\underline{|2n}}\right\} - (2n+1)\,\frac{\underline{|n}\,\underline{|n}}{\underline{|2n}}.$$

Ex. If $n=1$, $A=2$, a result which might have been inferred *à priori.*

If $n=2$, $A=2\frac{2}{3}$,

If $n=3$, $A=3\frac{1}{10}$,

If $n=\infty$, $A=3$, the case becomes identical with Ex. 3.

8. Suppose that when a man rolls a ball into a group of balls he is equally likely to hit each of them and *equally likely* to miss altogether, then if he first roll one ball and then roll another after it and so on until a ball has been hit, the average number of balls that he will roll will be e. (By Prop. II).

9. A bag contains n tickets marked with the letters A, B, C, ... The tickets are drawn one by one until A and B are both drawn. On an average $\frac{2}{3}(n+1)$ tickets will be drawn. (By Prop. VI).

10. A bag contains n tickets marked with different letters. The tickets are drawn one by one until r given tickets are all drawn. On an average $\dfrac{rn}{r+1}$ tickets will be drawn. (By Prop. VI).

Cambridge, *July*, 1871.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Feb. 8, Prof. Cayley, *Vice-President*, in the Chair. The Chairman mentioned that the President had made enquiry at the Home Office as to the mode of procedure requisite for obtaining a Charter for the Society, and that the matter would come on for consideration at the next subsequent meeting (March 14), when Members would have an opportunity of stating their views upon the desirability of incorporation. Mr. J. W. L. Glaisher, B.A., F.R.A.S., was elected a Member of the Society. Mr. T. Cotterill, M.A., gave an account of his paper, "On an Algebraical form, and the Geometry of its dual connection with a polygon, plane or spherical;" the Chairman, Dr. Hirst, and Prof. Clifford took part in the discussion on the paper.

R. TUCKER, M.A., *Hon. Sec.*

LECLERT'S THEOREM.

THE following paper contains an account of a theorem of great geometrical interest. As it relates primarily to floating bodies, it very properly found a place in the *Transactions of the Institution of Naval Architects*; but as these are not accessible to the general student of mathematics, I suggested to the Editors that it might be of interest to re-produce it in a journal of wider circulation.

The theorem virtually enables us to obtain the intrinsic elements of the curve enveloped by a chord which cuts off a segment of constant area from a given curve. It affords an elegant expression for the radius of curvature, which we can actually calculate, for any given direction of the normal to the chord, by means of quadratures.

The attempt to solve the general question involved by the direct application of coordinate geometry is defeated by the impossibility of distinguishing any two out of the several points in which a chord may cut a curved line. The escape from this difficulty by a proper choice of the variable elements makes it a very remarkable contribution to our geometrical knowledge.—C. W. MERRIFIELD.

ON CERTAIN THEOREMS RESPECTING THE GEOMETRY OF SHIPS.*

By EMILE LECLERT, Professor in the Ecole Impériale du Génie Maritime in Paris.

I.

The frequent use in Naval Architecture of considering a floating body, which heels over while retaining the same displacement, has led to the study of

1°. The surface which is the envelope of the planes which cut off displacements of a constant volume (V). This is called the *surface of flotation*, or more shortly, *surface F.*

2°. The surface which is the locus of the centres of buoyancy, or centres of upward pressure, of these volumes of equal displacement, called *surface of buoyancy*, or more shortly, *surface C.*

---

* From the *Transactions of the Institution of Naval Architects*, Vol. XI., p. 94, 1870.

The well-known memoir of M. Charles Dupin* has established the remarkable geometrical properties of these two kinds of surfaces.

In particular, for a vessel floating upright the radii of curvature r and R of the surface C—that is to say, the metacentric heights corresponding severally to an infinitesimal transversal or longitudinal inclination—have for their algebraical expressions

$$r = \frac{i}{V} = \frac{2}{3}\frac{\int y^3 dx}{V}, \quad R = \frac{I}{V} \ldots\ldots\ldots\ldots (1),$$

where i and I express the principal moments of inertia of the water section, and y the ordinates of the perimeter of this water section taken perpendicularly to the middle-line plane.† With regard to the surface F, Mons. Charles Dupin's investigation gives, as the expression for the transverse radius of curvature,

$$r_1 = \frac{\int y^2 \tan a ds}{\Omega} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where ds indicates an infinitesimal element of the perimeter of the water section, a the angle of the inclination of the side of the ship to the vertical direction, and Ω the area of the water section. The longitudinal radius of curvature R_1 is expressed by an analogous formula.

The formulæ (1) are easily calculated, and it is usual to compute them for all designs of ships for various values of V, but this is not done for formula (2). Nevertheless, a knowledge of the value of r_1, whether it be regarded as serving to determine the surface F, or whether it be considered as giving for small transverse inclinations a sufficient idea of this surface—and again, the knowledge of the value of R_1, with a view to questions relating to difference of trim—appear to be of more than merely theoretical interest to the Naval Architect. New expressions for the radii of curvature r_1 and R_1, capable of being easily calculated, may therefore prove useful in the preparation and consideration of designs for ships.

I shall, in the first place, establish my formulæ as consequences of the formula (2), and I shall afterwards give an independent demonstration of them.

---

* *Memoir on the Stability of Floating Bodies*, read before the Académie des Sciences, 10th January, 1814.

† This is the name given by naval architects to the plane of symmetry of this ship—the longitudinal vertical plane which contains the centre lines of the keel, masts, stempost, and sternpost.

II.

Consider (fig. 1) a floating body in its upright position. Let us take as known quantities Ω, V and i for a water section FL, defined in position by its distance z from a parallel plane taken as origin. We have

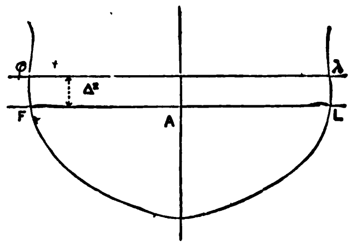
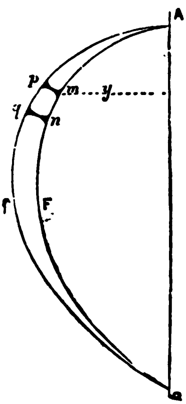


FIG. 1.

$$dV = \Omega dz.$$

Let the floating body be cut by a plane $\phi\lambda$ parallel to FL, and separated from it by a distance Δz which we shall bye-and-by make infinitesimal. Let the volume thus cut off from the floating body be called $V + \Delta V$, and the moment of inertia of $\phi\lambda$ about a longitudinal axis be called $i + \Delta i$.



Δi is simply the moment of inertia of the area comprised between the projections of the two planes of flotation on the water section. Let $mn = ds$ an elementary portion of the perimeter of FL, and let us draw mp and nq normal to this perimeter. If y be the ordinate and α the angle which the vertical through m makes with the surface of the floating body, we have

$$mp = \Delta z \tan \alpha.$$

Neglecting infinitesimals of the second order

$$\text{area } mnpq = \Delta z \tan \alpha ds,$$

$$\Delta i = \Sigma y^2 \Delta z \tan \alpha ds,$$

$$= \Delta z \Sigma y^2 \tan \alpha ds;$$

whence
$$\frac{di}{dz} = \int y^2 \tan \alpha ds.$$

Consequently, by virtue of equation (2)

$$r_1 = \frac{1}{\Omega} \frac{di}{dz},$$

$$= \frac{di}{dV} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

Before stopping to consider the conclusions which may be drawn from this relation, I shall transform it. Taken together with (1), it permits us to write

$$r_1 - r = \frac{di}{dV} - \frac{i}{V},$$

$$= \frac{Vdi - idV}{VdV},$$

which leads to $$r_1 = r + \frac{Vdr}{dV} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

I give the preference to this last formula, as expressing r_1 by quantities r, V, which are usually shown explicitly on our plans for various values of z.

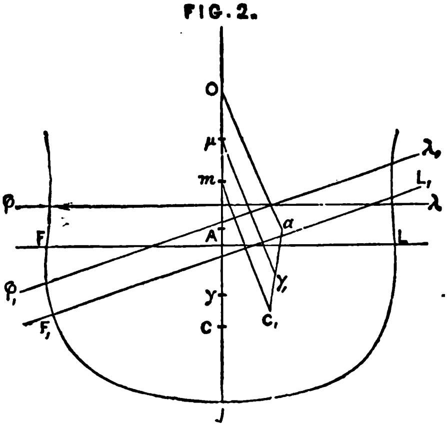
I now proceed to establish it by a direct method:

1°. For a volume of displacement V, let the upright water section be FL, the water section $F_1 L_1$ being inclined at an infinitesimal angle θ; let C and C_1 be the corresponding centres of buoyancy, m the metacentre, r the metacentric height Cm.

2°. Corresponding to a displacement $V + \Delta V$, the water sections $\phi\lambda$, $\phi_1\lambda_1$; let γ, γ_1 be the corresponding centres of buoyancy; μ the metacentre, and $r + \Delta r$ the metacentric height $\gamma\mu$.

Let us mark with the letters A, a the centres of buoyancy of the slices comprised between the upright and the inclined water sections respectively. The point of intersection O of the lines AO, aO perpendicular to FL, F_1L_1, tends to coincidence with the centre of transverse curvature of the flotation envelope of the displacements which are equal to V, as ΔV approaches zero; and at the same time the length AO tends to equality with r_1. This settled, it will be remarked that

FIG. 2.



γ_1 lies on the right line C_1a, and that since the right lines Oa, $\mu\gamma_1$, mC_1 are parallel, the point μ divides the distance mO in the same ratio as γ divides CA, and we thus obtain the set of equations

$$\frac{Om}{m\mu} = \frac{CA}{C\gamma} = \frac{Om - CA}{m\mu - C\gamma} = \frac{V + \Delta V}{\Delta V}.$$

We have also the following identities: firstly

$$Om - CA = (Om + Am) - (CA + Am) = OA - Cm = OA - r;$$

and secondly

$$m\mu - C\gamma = (\gamma m + m\mu) - (C\gamma + \gamma m) = \gamma\mu - Cm = \Delta r.$$

The two last terms of the set of equations among the ratios thus enable us to write down

$$\frac{OA - r}{\Delta r} = \frac{V + \Delta V}{\Delta V};$$

or

$$OA - r = (V + \Delta V)\frac{\Delta r}{\Delta V}.$$

This becomes at the limit

$$r_1 - r = V\frac{dr}{dV},$$

which agrees with formula (4).

The method of calculation is that which is usual in such cases, to take for $\frac{dr}{dV}$ the ratio $\frac{\Delta r}{\Delta V}$ relatively to two consecutive groups of simultaneous values of r and V. It is very easy, moreover, to draw a curve of which V is the abscissa and r the ordinate. Theoretically, formula (4) gives, with the help of this curve, a very simple construction for r_1, whatever be the scales on which V and r are set off. In any case, such a curve will be a useful graphical auxiliary to the computation above mentioned.

III.

In what goes before, I have only considered the vessel as being upright in its vertical position. If it were inclined, the expressions corresponding to r, r_1, would be those for the radii of curvature of the cylinders with horizontal generating lines circumscribed to the surfaces F, C.* The right sections

* See Note at end of Paper.

of these cylinders are usually called the *envelope of flotation* and *curve of buoyancy*, when we are examining a vessel's heeling, or when we wish to plot its metacentric evolute for transverse inclinations.

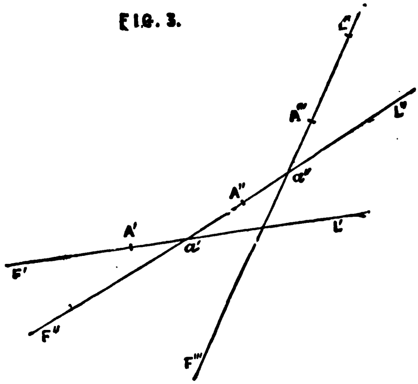
The use of the radius of curvature r_1 abbreviates the plotting: in fact let $F'L'$ (fig. 3) be the trace of a water section cutting off a volume V, A' the centre of gravity of the water section, r'_1 the radius of curvature of the flotation envelope at A': a water section $F''L''$ of the same displacement, making a small angle θ with $F'L'$ will cut $F'L'$ at a point a' such that

$$A'a' = r'_1 \tan \tfrac{1}{2}\theta.$$

We can thus pass in succession from the upright position to any series of inclined positions, by calculating for each of them, such as $F'L'$ (fig. 3), the position of its centre of gravity A'; the radius of curvature of the curve of buoyancy (the metacentric height for an infinitesimal angle measured from $F'L'$), or

$$r' = \frac{i'}{V}.$$

FIG. 3.



The radius of curvature of the envelope of flotation is

$$r'_1 = r' + V\frac{dr'}{dV}.$$

The ratio $\frac{\Delta r'}{\Delta V}$, used to get the coefficient $\frac{dr'}{dV}$, will be found by means of auxiliary water sections (one or two) parallel to $F'L'$, and very near to it.

The last inclined section of the series may be verified by a direct calculation of its volume by one of the ordinary methods. If a correction has to be made, it will be easy to introduce the necessary corrections into the intermediate sections by means of interpolations obtained with the help of the auxiliary water sections. They may even be settled by the simple consideration of the continuous character of the variation of the lengths $A'a'$, $a'A''$, $A''a''$, ... in consequence of the equality of the angles θ which the successive

water sections $F'L'$, $F''L''$, $F'''L'''$, ... make with one another.

Besides, there are not wanting methods of tracing the metacentric evolute by means of the calculated lengths of the radius of curvature r, r', &c., of the curve of buoyancy. If it became necessary to plot the flotation envelope and the metacentric evolute for various values of V, it would be easy to give the extension necessary for this purpose to the artifice of introducing the auxiliary water sections which we have just mentioned.

IV.

Finally, I wish to point out that the leading formulæ

$$r_1 = \frac{di}{dV}, \quad r_1 = r + V\frac{dr}{dV},$$

and the corresponding formulæ for the longitudinal radius of curvature

$$R_1 = \frac{dI}{dV}, \quad R_1 = R + V\frac{dR}{dV},$$

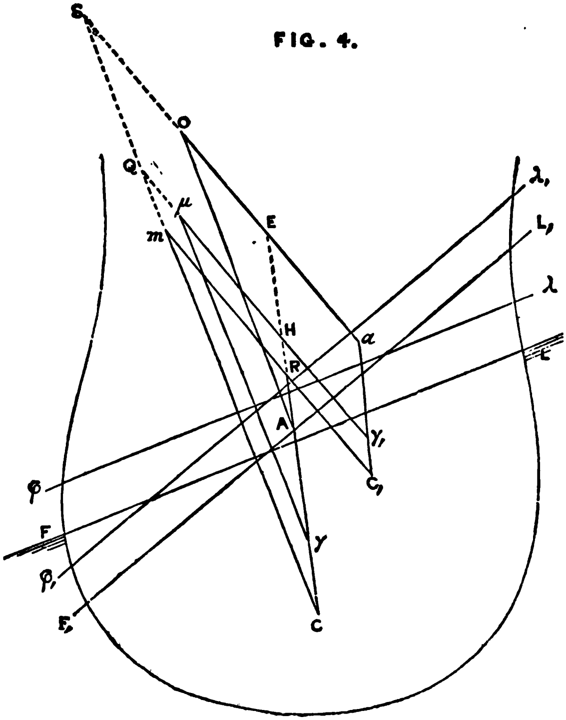
are the algebraical expression of the theorems which explicitly connect the elements of curvature of the surface F, with those of the surface C. Apart from their special application to Naval Architecture, it appears to me that these theorems lead, as a question of pure analysis, to some interesting considerations relating to the geometrical interdependence of these two classes of surfaces.

NOTE. The direct proof of formula (4), as thus extended, is easily established, when the initial water section is inclined, as is the case with FL (fig. 4). From this point of view fig. 4 is a repetition of fig. 2: the centres of buoyancy, or of upright pressure C, γ, A, are in one right line, as are also the points C_1, γ_1, a. The directions of the different pressures considered, give rise to two series of three parallel right lines Cm, $\gamma\mu$, AO, and C_1m, $\gamma_1\mu$, aO. These last intersect the line CA produced in K, H, E, and the line Cm produced in Q and S.

This stated, by similar reasoning to that which we used in the case of fig. 2, we obtain the following series of equations:

$$\frac{EK}{KH} = \frac{CA}{C\gamma} = \frac{EK - CA}{KH - C\gamma} = \frac{V + \Delta V}{\Delta V} \text{.........}(a).$$

FIG. 4.



But we have identically

$$EK - CA = (EK + KA) - (CA + AK) = EA - CK,$$

$$KH - C\gamma = (\gamma K + KH) - (\gamma K + C\gamma) = \gamma H - CK;$$

whence
$$\frac{EK - CA}{KH - C\gamma} = \frac{EA - CK}{\gamma H - CK} \quad \ldots\ldots\ldots\ldots\ldots\ldots (b).$$

Again, from the figure

$$\frac{EA}{OA} = \frac{EC}{CS}, \quad \frac{CK}{Cm} = \frac{CE}{CS}, \quad \frac{\gamma H}{\gamma\mu} = \frac{CH}{CQ},$$

the three second members being clearly equal, the first must also be, and we get

$$\frac{EA}{OA} = \frac{CK}{Cm} = \frac{\gamma H}{\gamma\mu};$$

whence it follows that we can write

$$\frac{EA-CK}{\gamma H-CK}=\frac{OA-Cm}{\gamma\mu-Cm}.$$

Consequently, by virtue of (*a*) and (*b*) we have

$$\frac{OA-Cm}{\gamma\mu-Cm}=\frac{V+\Delta V}{\Delta V};$$

that is to say
$$\frac{OA-r}{\Delta r}=\frac{V+\Delta V}{\Delta V}.$$

Then at the limit
$$r_1-r=\frac{Vdr}{dV}.$$

Q.E.D.

TWO PROBLEMS IN CALCULUS OF VARIATIONS.

By *M. M. U. Wilkinson, M.A.*

Of the two problems discussed in this paper the first exemplifies that it is very seldom that a simply continuous solution exists for a problem in calculus of variations, and the second exemplifies the method of finding the greatest maxima and the least minima of a large class of integrals.

I. Having given
$$u\equiv\int_{-h}^{+h}zdx;$$

$$z\equiv 12a^4-3a^2x^2-6axy^2+4(a+x)y^3-3y^4;$$

to determine what function y is of x when u is a maximum, h is supposed greater than $2a$.

Here
$$N\equiv-12y(y-a)(y-x),$$

and
$$\frac{dN}{dy}=-12ax+24(a+x)y-36y^2.$$

So we must have either

$$y=I(x),\quad y=a+I(x),\quad\text{or}\quad y=x+I(x);$$

but not the first if x is negative, nor the second if x is greater than a, nor the third if x is positive and less than a. So there is no simply continuous maximum solution. For the maximum maximorum

$$\begin{aligned} y &= x + I(x), \text{ from } x = -h \text{ to } x = -a, \\ y &= a + I(x), \text{ from } x = -a \text{ to } x = \frac{a}{2}; \\ y &= 0 + I(x), \text{ from } x = \frac{a}{2} \text{ to } x = 2a, \\ y &= x + I(x), \text{ from } x = 2a \text{ to } x = h, \end{aligned}$$

as may be seen by inspecting which of the three

$$12a^4 - 3a^2x^2, \ 13a^4 - 2a^3x - 3a^2x^2, \ 12a^4 - 3a^2x^2 - 2ax^3 + x^4,$$

is the greatest.

The geometrical significance of the problem is this: suppose a point is required to travel upon a surface, (1) so that the projection of its path, the axis of x, and ordinates at the extremities of the path, upon the plane of xz, should enclose the greatest possible area, to find the equation to the projection of this path upon the plane of xy. Calling projections on the plane of xy of sections of the surface, throughout which z is constant, maps, and projections in the plane of xz of sections of the surface, throughout which y is constant, landscapes, our problem is simply to trace on the map the line corresponding to the sky line on the landscape. Compare the accompanying map and landscape (figs. 22 and 23). The point must travel along the sky line, and on arriving at T_1, T_2, T_3, (the point at which we cannot have, $I(x) = 0$), must pass from one ridge to another in the plane parallel to the plane of yz.

II. Having given

$$u \equiv \int_0^h p^3 (p - a)\, dx, \quad k = \int_0^h p dx,$$

to determine under what circumstances u is a minimum. We must consider the minima values of

$$v \equiv \int_0^h (p^4 - ap^3 + \lambda p)\, dx,$$

whence

$$P \equiv 4p^3 - 3ap^2 + \lambda,$$

$$\frac{dP}{dp} = 12p^2 - 6ap,$$

so that, for a minimum, p cannot range between 0 and $\frac{a}{2}$, and, when $\lambda > 0 < \frac{a^3}{4}$, there exist two values for p, both of which make $P=0$, and one is negative, and the other greater than $\frac{a}{2}$. Among all these minima to find the minima minimorum we must apply Differential Calculus methods. The investigation can conveniently be conducted by making u a minimum, p_1, p_2 varying, and

$$u = (p_1^4 - ap_1^3)\,\xi + (p_2^4 - ap_2^3)(h - \xi);$$

$$\xi = \frac{k - hp_2}{p_1 - p_2}, \quad h - \xi = \frac{p_1 h - k}{p_1 - p_2},$$

so that

$$u = \frac{k(p_1^4 - ap_1^3 - p_2^4 + ap_2^3) - hp_1p_2(p_1^3 - ap_1^2 - p_2^3 + ap_2^2)}{p_1 - p_2},$$

whence $\quad 0 = 3p_1^4 - 2ap_1^3 - 4p_1^3p_2 + 3ap_1^2p_2 + p_2^4 - ap_2^3,$

$$0 = 3p_2^4 - 2ap_2^3 - 4p_2^3p_1 + 3ap_2^2p_1 + p_1^4 - ap_1^3,$$

$$4p_1^3 - 3ap_1^2 = 4p_2^3 - 3ap_2^2 = -\lambda$$

$$= \frac{p_1^4 - ap_1^3 - p_2^4 + ap_2^3}{p_1 - p_2},$$

equations which give

$$p_1 = \frac{1 + \sqrt{3}}{4}\,a, \quad p_2 = \frac{1 - \sqrt{3}}{4}\,a.$$

The condition that p_2, $\frac{k}{h}$, p_1 are in order must be satisfied. If this condition is not satisfied, we cannot get a less minimum than that given by the line $y = \frac{k}{h}\,x$.

ON A PENULTIMATE QUARTIC CURVE.

By Professor *Cayley*.

I HAVE had occasion to consider with some particularity the form of a curve about to degenerate into a system of multiple curves; a simple instance is a trinodal quartic curve about to degenerate into the form $x^2y^2=0$, or say a "penultimate" of $x^2y^2=0$. To fix the ideas take x, y, z to denote the perpendiculars on the sides of an equilateral triangle, altitude $=1$ (so that $x+y+z=1$), and let the curve be symmetrical in regard to the coordinates x, y, its equation being thus

$$(a,\ a,\ 1,\ f,\ f,\ h)(x,\ y,\ z)^2=0,$$

where a, f, h are ultimately all indefinitely small in regard to unity: to diminish the number of cases I further assume

$$a=+,\ f \text{ and } h=-$$

$$h^2>a^2,\ \text{ that is } a+h=-$$

$$f>a,\quad ,,\quad ,,\quad \sqrt{(a)}+f=-$$

but I do not in the first instance take a, f, h to be indefinitely small. Then if $-f$ is not too large, the curve is as shewn in fig. 24,* viz. it is a triloop curve, with two horizontal double tangents, 3 touching the curve in two real points, 4 touching it in two imaginary points. Imagine $-f$ increased, the new curve will have the same general form, intersecting the first curve at A and B but touching it at C, viz. it will pass inside the loop C but outside the loops A, B; and outside the remainder of the curve; and the 4 will also move downwards as shewn. New position of 4 will be below the first position.

Supposing that a, h have given values, and that $-f$ continually is increased in regard to $\sqrt{(a)}$, two things may happen. 1°. The double tangent 3 may move down to $z=-\infty$ (the lower loops lengthening out, and finally becoming each of them a pair of parabolic branches parallel at infinity); and then reappearing at $z=+\infty$ again move downwards, each loop becoming in this case a pair of hyperbolic branches touching two asymptotes at $z=-\infty$, and

then again on the opposite sides thereof at $z = +\infty$, and coming down as a single branch to touch the double tangent 3 which is now above 4. 2°. The double tangent 4 may come to coincide with the horizontal tangent 2; at the instant of coincidence being a tangent of four-pointic contact; and becoming afterwards (being as before above 2) an ordinary double tangent with two real points of contact; viz. instead of a simple loop at C we have a heart-shaped loop.

But to investigate whether 1° and 2° actually happen, and in what order of succession, we require the expressions of z for the several lines in question; we find, without difficulty,

$$\begin{array}{llll} \text{for line } 1 & z_1 = \dfrac{1}{1+2\lambda_1} & \text{where} & \lambda_1 = -2f + \sqrt{\{4f^2 - 2(a+h)\}}, \\ \quad ,, \quad 2 & z_2 = \dfrac{1}{1-2\lambda_2} & \quad ,, & \lambda_2 = 2f + \sqrt{\{4f^2 - 2(a+h)\}}, \\ \quad ,, \quad 3 & z_3 = \dfrac{1}{1-2\lambda_3} & \quad ,, & \lambda_3 = \dfrac{a-h}{2\{-\sqrt{(a)} - f\}}, \\ \quad ,, \quad 4 & z_4 = \dfrac{1}{1-2\lambda_4} & \quad ,, & \lambda_4 = \dfrac{a-h}{2\{\sqrt{(a)} - f\}}, \end{array}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are all positive. Observe that in the limiting case $-f = \sqrt{(a)}$, where, instead of the loops at A, B, we have cusps, z_1, z_2, and z_4 are (in general) positive; $\lambda_3 = \infty$, and therefore $z_4 = 0$; that is, the line 3 coincides with AB, ceasing to be a double tangent; there is in this case the one double tangent 4.

1°. z_3 becomes infinite for $1-2\lambda_3=0$; that is, $a-h=-\sqrt{(a)}-f$, or $-f = \sqrt{(a)} + (a-h)$; viz. for $-f = \sqrt{(a)} + (a-h) - \varepsilon$, we have $z_3 = -\infty$, and for $-f = \sqrt{(a)} + (a-h) + \varepsilon$, we have $z_3 = +\infty$.

2°. The lines 4 and 2 will coincide if

$$\lambda_4 \left[= \frac{a-h}{2\{\sqrt{(a)} - f\}} \right] = 2f + \sqrt{\{4f^2 - 2(a+h)\}},$$

that is $$\lambda_4(\lambda_4 - 4f) = -2(a+h),$$

or substituting for λ_4 its value

$$(a-h)[(a-h) - 8f\{\sqrt{(a)} - f\}] + 8(a+h)\{\sqrt{(a)} - f\}^2 = 0,$$

that is $\{3a + h - 4f\sqrt{(a)}\}^2 = 0$, or $4f\sqrt{(a)} = 3a + h$,

(observe that f having been assumed negative, this implies $-h > 3a$). That is, $3a+h$ being $=-$, but not otherwise the double tangent 4, will for the value $-f=\dfrac{-3a-h}{4\sqrt{(a)}}$, come to coincide with the line 2; and for any greater value of $-f$ will be as before above line 2, (being in this case an ordinary double tangent with real points of contact) as appears from the form, $U^2=0$, of the foregoing equation for the determination of f.

The passage of the line 3 to infinity, and the coincidence of the lines 4 and 2 may take place for the same value of f, viz. this will be the case if

$$\sqrt{(a)}+(a-h)=\frac{-3a-h}{4\sqrt{(a)}},$$

that is $7a+4\sqrt{(a)}+h\{1-4\sqrt{(a)}\}=0$ or $-h=\dfrac{a\{7+4\sqrt{(a)}\}}{1-4\sqrt{(a)}}$,

or, a being small, for the value $-h=7a$ approximately. If $-h$ is less than the above value, then $\dfrac{-3a-h}{4\sqrt{(a)}}$ is less than $\sqrt{(a)}+a-h$, or $-f$ increasing from $\sqrt{(a)}$, the coincidence of the lines 2, 4 takes place before the line 3 goes off to infinity: contrarywise if $-h$ is greater than the above value.

In any form of the curve (*i.e.* whatever be the value of f in regard to a, h) if we imagine a, h indefinitely diminished, the lines 1, 2 and 4 will continually approach C, and the curve will gather itself up into certain definite portions of the lines $x=0$, $y=0$. Thus any secant through A (not being indefinitely near to the line AC) which meets the curve in real points will meet it in two points tending to coincide at the intersection of the secant with the line $x=0$; analytically there are always two intersections real or imaginary which (the secant not being indefinitely near the line AC) tend to coincide at the intersection of the secant with the line $x=0$; and we thus see how we ultimately arrive at the line $x=0$ twice repeated; and similarly for the line $y=0$.

* The figure is drawn with very small values of a, f, h in order to exhibit as nearly as may be one of the penultimate forms of the curve, but this is not in anywise assumed in the reasoning of the text. Observe in the figure that the points A, B are ordinary double points, there are at each of them two distinct tangents inclined at a small angle to each other.

PROOF OF EUCLID II. 8.

By *C. Taylor, M.A.*

1. It is evident that four rectangles of sides a, b can be fitted symmetrically, as in fig. 25, about the square of $a-b$, and that the area thus made up is the square of $a+b$.

Therefore $$(a+b)^2=(a-b)^2+4ab.$$

The fault of the construction in Euc. II. 8 is that it is unsymmetrical and does not shew the rectangles themselves, but only an equal gnomon.

Owing to the unattractiveness of the proof the proposition is commonly omitted, and the relation

$$(a+b)^2-(a\sim b)^2=4ab$$

comes to be looked upon as a formula belonging exclusively to algebra, when its geometrical proof is in reality quite as satisfactory as that of Euc. II. 4, and indeed more so, inasmuch as the construction may be made more symmetrical.

2. A perfectly symmetrical figure results from producing all the sides of the inner square both ways. Thus the large square is divided into a central square, four corner squares, and four rectangles. If a side of the central square be called x, and a side of a corner square y, there results

$$(x+2y)^2=x^2+4xy+4y^2.$$

NOTE ON A FORMER PAPER.

By *R. W. Genese, B.A.*

Since writing my note on the method given by M. Chasles another method has occurred to me which is too simple for me to expect it to be new; however, as I know of no text-book in which the matter is treated, I venture to publish it.

If P (fig. 26) be a point of trisection of the arc AB, the chord BP is twice the sine of half BP or $2PM$, where PM is the perpendicular from P on OA.

Therefore P lies on a hyperbola of eccentricity Q, focus B, and directrix OA.

THEOREMS RELATING TO THE CENTRE OF PRESSURE.

By *Arthur Hill Curtis, LL.D.*

In a paper published in *The Messenger*, Vol. II., p. 250, I have proved that if a triangle, of which the sides are a, b, c, the angles opposite to these sides, respectively, A, B, C, and whose area is N, be immersed in a homogeneous liquid, and if the perpendicular distances of the centre of pressure of the triangle from the sides a, b, c, be denoted by x, y, z, while the depths, to which the angles are immersed, are represented by h_1, h_2, h_3, then x, y, z will be given by the formulæ:

$$x=\frac{N}{2a}\left\{1+\frac{h_1}{h_1+h_2+h_3}\right\},$$

$$y=\frac{N}{2b}\left\{1+\frac{h_2}{h_1+h_2+h_3}\right\},$$

$$z=\frac{N}{2c}\left\{1+\frac{h_3}{h_1+h_2+h_3}\right\}.$$

For the purposes of the present paper I shall slightly alter the form of these equations. Let H_1, H_2, H_3 be the depths, to which the middle points of the three sides, a, b, c, are immersed, then $h_2+h_3=2H_1$, $h_3+h_1=2H_2$, $h_1+h_2=2H_3$; therefore

$$x=\frac{N}{a}\left\{\frac{H_2+H_3}{H_1+H_2+H_3}\right\},$$

$$y=\frac{N}{b}\left\{\frac{H_1+H_3}{H_1+H_2+H_3}\right\},$$

$$z=\frac{N}{c}\left\{\frac{H_1+H_2}{H_1+H_2+H_3}\right\}.$$

It is plain that these formulæ, or those from which they have been deduced, not only determine the centre of pressure of a triangular area sunk in a homogeneous liquid under given conditions, but will serve also to determine the ratios of the depths, to which the angles, or the middle points of sides, must be immersed in order that the centre of pressure may coincide with any given point situated within the triangle. It is the *ratios* only, and not the depths themselves, which can be determined, because these three equations are not independent, being connected by the trilinear relation

$ax + by + cz = 2N$, and are really equivalent to *two* independent equations. To determine the depths themselves some other condition must be added; such would be supplied, for example, if the total pressure on the area were given, or, which is equivalent to this condition, the depth of the centre of gravity of the area, or more generally the depth of some given point situated within the area. It might be expected that relations more or less symmetrical would exist between H_1, H_2, H_3, if the centre of pressure coincided with a point symmetrically placed with regard to the three sides of the triangle, or any two of them, and I propose in the present paper to consider a few cases of this kind. The points which most obviously suggest themselves are—

(1) The centre of gravity,

(2) The centre of the circle inscribed,

(3) The centre of the circle circumscribed,

(4) The intersection of the three perpendiculars from the angles on the opposite sides,

(5) The centre of any one of the three inscribed squares,

(6) The centre of the circle, which passes through the three middle points of sides, and the feet of the three perpendiculars from the angles on the opposite sides.

(1) If the centre of pressure coincide with the centre of gravity, it is evident *à priori* that the plane of the triangle must be horizontal, this also follows easily from the formulæ, for in this case

$$x = \frac{2N}{3a}, \quad y = \frac{2N}{3b}, \quad z = \frac{2N}{3c};$$

therefore $$\frac{H_2 + H_3}{H_1 + H_2 + H_3} = \frac{2}{3} = \frac{H_1 + H_3}{H_1 + H_2 + H_3};$$

therefore $$H_1 = H_2 = H_3.$$

(2) If the centre of pressure coincide with the centre of the inscribed circle, then

$$H_1 : H_2 : H_3 :: \cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2}.$$

For obviously $x = y = z$; therefore

$$H_2 + H_3 : H_1 + H_3 : H_1 + H_2 :: a : b : c;$$

therefore

$$H_1 : H_2 : H_3 :: b + c - a : a + c - b : a + b - c :: \cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2}.$$

(3) If the centre of pressure coincide with the centre of the circumscribed circle, then

$$H_1 : H_2 : H_3 :: \tan A : \tan B : \tan C.$$

In this case it is evident that if R denote the radius of the circumscribed circle $x = R\cos A$, $y = R\cos B$, $z = R\cos C$; therefore

$$H_2 + H_3 : H_1 + H_3 : H_1 + H_2 :: a\cos A : b\cos B : c\cos C$$
$$:: \sin 2A : \sin 2B : \sin 2C;$$

therefore $H_1 : H_2 : H_3 :: \sin 2B + \sin 2C - \sin 2A$

$$: \sin 2A + \sin 2C - \sin 2B : \sin 2A + \sin 2B - \sin 2C,$$

or, by an easy reduction, $H_1 : H_2 : H_3 :: \tan A : \tan B : \tan C$.

(4) If the centre of pressure coincide with the intersection of the three perpendiculars from the angles on the opposite sides, we must have

$$H_1 : H_2 : H_3 :: 1 - 2\cot B\cot C : 1 - 2\cot A\cot C : 1 - 2\cot A\cot B.$$

If p_1 denote the perpendicular let fall from A on a, it is geometrically evident that

$$x = p_1 \cot B\cot C = \frac{2N}{a}\cot B\cot C;$$

therefore $$\frac{2N}{a}\cot B\cot C = \frac{N}{a}\left\{\frac{H_2 + H_3}{H_1 + H_2 + H_3}\right\};$$

therefore $$2\cot B\cot C = \frac{H_2 + H_3}{H_1 + H_2 + H_3},$$

similarly $$2\cot A\cot C = \frac{H_1 + H_3}{H_1 + H_2 + H_3};$$

therefore, as $\cot A\cot B + \cot B\cot C + \cot A\cot C = 1$,

$$H_1 : H_2 : H_3 :: 1 - 2\cot B\cot C : 1 - 2\cot A\cot C : 1 - 2\cot A\cot B.$$

(5) If the centre of pressure coincide with the centre of the inscribed square standing on the side a, we must have

$$H_1 : H_2 : H_3 :: 1 : \cot B : \cot C.$$

If s denote the side of the inscribed square, it is geometrically evident that

$$x = \frac{s}{2},\quad y = \frac{s}{2}(\cos C + \sin C),\quad z = \frac{s}{2}(\cos B + \sin B);$$

comparing these values with those given by the formulæ, we get

$$\cos C + \sin C = \frac{y}{x} = \frac{a\,(H_1 + H_3)}{b\,(H_2 + H_3)},$$

or $\dfrac{H_1 + H_3}{H_2 + H_3} = \dfrac{b}{a}\,(\cos C + \sin C)$

$$= \frac{b\,(\cos C + \sin C)}{b\,\cos C + c\,\cos B} = \frac{\sin B\,(\cos C + \sin C)}{\sin B\,\cos C + \sin C\,\cos B} = \frac{1 + \cot C}{\cot B + \cot C};$$

similarly $$\frac{H_1 + H_2}{H_2 + H_3} = \frac{1 + \cot B}{\cot B + \cot C};$$

therefore $$H_1 : H_2 : H_3 :: 1 : \cot B : \cot C.$$

(6) If the centre of pressure coincide with the centre of the circle, known as the *nine-point circle*, which passes through the three middle points of sides, and also through the feet of the three perpendiculars let fall from the three angles on the opposite sides, we must have

$$H_1 : H_2 : H_3 :: \sin 2A : \sin 2B : \sin 2C.$$

It is evident, geometrically, that the distance of this point from the line joining the middle points of b and c is $\frac{a}{4}\cot A$, and, as this line is parallel to a, and at a distance from it equal to $\frac{p_1}{2}$, we have

$$\frac{p_1}{2} - \frac{a}{4}\cot A = x = \frac{p_1}{2}\left(\frac{H_2 + H_3}{H_1 + H_2 + H_3}\right),$$

or $$\frac{a}{4}\cot A = \frac{p_1}{2}\left(1 - \frac{H_2 + H_3}{H_1 + H_2 + H_3}\right) = \frac{p_1}{2}\,\frac{H_1}{H_1 + H_2 + H_3},$$

but $$p_1 = \frac{a\,\sin B\,\sin C}{\sin A};$$

therefore $$\frac{H_1}{\sin 2A} = \frac{H_1 + H_2 + H_3}{4\,\sin A\,\sin B\,\sin C} = \frac{H_2}{\sin 2B} = \frac{H_3}{\sin 2C},$$

or $$H_1 : H_2 : H_3 :: \sin 2A : \sin 2B : \sin 2C.$$

By comparing the results obtained in (3) and (4) it can easily be shewn that if a triangular area be so immersed in a homogeneous liquid, that its centre of pressure coincides with the intersection of the three perpendiculars let fall from the angles on the opposite sides, then the centre of pressure of the triangular area enclosed by the three lines joining the middle points of the sides of the original triangle will coincide with the centre of its own circumscribing circle.

By the condition of the question we have from (4)

$$H_1 : H_2 : H_3 :: 1-2\cot B\cot C : 1-2\cot A\cot C : 1-2\cot A\cot B,$$

or, since $1-\cot A\cot B-\cot A\cot C-\cot B\cot C=0$,

$$\frac{H_2+H_3}{2} : \frac{H_3+H_1}{2} : \frac{H_1+H_2}{2} :: \cot B\cot C : \cot A\cot C :: \cot A\cot B,$$

$$:: \tan A :: \tan B :: \tan C,$$

but as A, B, C, are the angles of the derived triangle as well as of the original, and as the depths, to which the middle points of the former are immersed, are

$$\frac{H_2+H_3}{2}, \frac{H_3+H_1}{2}, \frac{H_1+H_2}{2},$$

the above relation is by (3) that required in order that its centre of pressure should coincide with the centre of its circumscribed circle, and the theorem is established.

Again, by comparing (3) and (6) we learn that, if a triangular area be so immersed in a homogeneous liquid that its centre of pressure coincides with the centre of its circumscribed circle, then the centre of pressure of the triangular area enclosed by the three lines joining the middle points of the sides of the original triangle will coincide with the centre of its own *nine-point circle.*

By the condition of the question we have from (3)

$$H_1 : H_2 :: H_3 :: \tan A : \tan B : \tan C,$$

from which it may be easily proved that

$$\frac{H_2+H_3}{2} : \frac{H_3+H_1}{2} : \frac{H_1+H_2}{2} :: \sin 2A : \sin 2B : \sin 2C;$$

it is plain therefore, as above, that the condition required in (6) is fulfilled.

By combining these two results we learn that, if a triangular area be so immersed in a homogeneous liquid that its centre of pressure coincides with the intersection of the three perpendiculars let fall from its angles on the opposite sides, then the centre of pressure of the triangular area enclosed by the three lines joining the middle points of its sides will coincide with the centre of the circle circumscribed to the latter, and that the centre of pressure of the triangular area similarly derived from it will coincide with the centre of its own *nine-point circle.*

The following theorem will, in several cases, be found

useful in determining the centre of pressure of a plane area immersed in a homogeneous liquid.

If the centre of gravity of a plane area, N, bisect any line, AB, in its plane, and in the case in which one extremity, A, of this line is in the surface of a homogeneous liquid, in which the entire area is immersed, the centre of pressure cut it in C so that $AC : CB :: m : n$, then, if by a motion of translation the area be brought into a position such that A and B are at depths, h_1, h_2 respectively, the centre of pressure, C', will cut AB so that

$$AC' : C'B :: mh_2 + nh_1 :: mh_1 + nh_2.$$

This may easily be established as follows:

In the second position of the area, the normal pressure to which any small element of the area is exposed, is equal to the weight of the quantity of the liquid, which would be contained in the cylinder, whose base is the area, and whose height is the depth of the area below the surface. Conceive a horizontal plane through A, and let z be the depth of the elementary area below this plane, its depth below the surface of the liquid will then be $h_1 + z$, and supposing the whole area divided into a number of indefinitely small and equal elements, the actual pressure on each of them will be the sum of two; the first due to the constant height h_1, the other to the variable height z, the first of these extended to the whole area will be applied at its centre of gravity, and its magnitude will be represented by wNh_1, w being the weight of the unit of volume of the liquid, the second similarly extended will be the pressure, to which the area would be exposed, if the surface of the liquid were lowered to A, it is therefore applied at C, while its magnitude is represented by $wN\bar{z}$, $\bar{z}$ being the depth of the centre of gravity of the area below the horizontal plane through A, and therefore equal to $\frac{h_2 - h_1}{2}$. Denoting these two parts of the total pressure by P_1, P_2, we have $P_1 = wNh_1$, $P_2 = wN\,\frac{(h_2 - h_1)}{2}$, while, if AB be denoted by a, the distances of the points of application of P_1, P_2 from A will be respectively $\frac{a}{2}$, and $\frac{m}{m+n}\,a$; if $\bar{x}$, therefore, be the distance of A from the point of application of the resultant of P_1 and P_2, which is plainly the centre of pressure, we have

$$(P_1 + P_2)\,x = P_1\,\frac{a}{2} + P_2\,\frac{m}{m+n}\,a,$$

or $$\frac{h_1+h_2}{2}x=h_1\frac{a}{2}+\frac{h_2-h_1}{2}\frac{m}{m+n}a=\frac{mh_2+nh_1}{m+n}\frac{a}{2};$$

therefore $\dfrac{x}{a}=\dfrac{mh_2+nh_1}{(m+n)(h_1+h_2)}$; therefore $\dfrac{x}{a-x}=\dfrac{mh_1+nh_2}{mh_1+nh_2}$.

For an example of the utility of this formula let it be proposed to determine the centre of pressure of a parallelogram immersed in a homogeneous liquid, so that one diagonal is horizontal, and that the extremities of the other AB are sunk to depths h_1, and h_2,

It is plain that the centre of gravity of the parallelogram bisects the diagonals, and it can easily be proved that, when h_1 is zero, the centre of pressure cuts the diagonal AB in C so that $AC : CB :: 7 : 5$, therefore the centre of pressure in the case supposed in the problem cuts AB in C' so that $AC' : C'B :: 7h_2 + 5h_1 : 7h_1 + 5h_2$.

Again, it is easily shewn that, if a parallelogram be immersed in a homogeneous liquid so that one side is situated in the surface of the liquid, the centre of pressure will cut the line joining the middle points of the two horizontal sides in the ratio $2 : 1$; hence we infer that, if a parallelogram be immersed in a homogeneous liquid in such a way that two sides are horizontal, and at depths below the surface represented by h_1, h_2, the centre of pressure will cut the line joining the middle points of the two horizontal sides in the ratio $2h_2 + h_1 : 2h_1 + h_2$.

Again it can be proved, without difficulty, that if an elliptic area be so immersed in a homogeneous liquid that the surface of the liquid touches the ellipse, the centre of pressure cuts the diameter AB passing through the point of contact, A, in C, so that $AC : CB :: 5 : 3$; hence we infer that, if an elliptic area be so immersed in a homogeneous liquid that A and B, the highest, and lowest points on the ellipse are at depths h_1, h_2, the centre of pressure will cut the diameter AB in a point C' so that

$$AC' : C'B :: 5h_2 + 3h_1 : 5h_1 + 3h_2.$$

Again, it can be proved, in an elementary way, that if the area bounded by a regular hexagon be immersed in a homogeneous liquid so that one side is in the surface of the liquid, the centre of pressure cuts the line joining the middle point of this side to the middle point of the parallel side in the ratio $23 : 13$; hence, if a regular hexagon be immersed in a liquid so that two parallel sides are horizontal and at depths

h_1, h_2, the centre of pressure cuts the line joining the middle points of these two sides in the ratio $23h_2 + 13h_1 : 23h_1 + 13h_2$.

If, instead of supposing the centre of gravity of the area to bisect the line AB, we suppose it to cut it in the ratio $m' : n'$, all else being as in the enunciation of the former theorem it can be shewn in a similar manner that

$$AC' : C'B :: m'(mh_2 + nh_1) : m'nh_2 + \{(m+n)n' - m'n)\} h_1,$$

a result which though not so symmetrical, or so generally applicable as the preceding, may sometimes be useful; for example, it can be proved that, if the area bounded by a cycloid, the radius of whose generating circle is a, is immersed in a homogeneous liquid in such a way that its vertex is in the surface of the liquid, and its base horizontal, the centre of pressure will divide the axis of the cycloid in the ratio $59 : 25$; hence, as the centre of gravity of the cycloidal area divides the axis of the cycloid in the ratio $7 : 5$, we infer that, if a cycloidal area be immersed in a homogeneous liquid in such a way that its base is horizontal, and its vertex and base at depths h_1, h_2, the centre of pressure will cut the axis in the ratio $25h_1 + 59h_2 : 25h_2 + 35h_1$.

Queen's College, Galway,
March 14*th*, 1872.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, March 14, W. Spottiswoode, Esq., F.R.S., *President*, in the Chair, Mr. W. Paice, M.A., London, was proposed for election.

On the proposal of the President it was agreed that application should be made to the Council Office for the grant of a Charter, the draft of which had been drawn up by Prof. Cayley. A vote of thanks was passed to Mr. S. M. Drach for his present to the Society of the *Opera Mathematica of Vieta* (edited by Schooten, 1646), and the *Mecanicorum Liber of Guidi ubaldi* (1615). Prof. Clifford gave a detailed account of his paper "On a new expression of Invariants and Covariants by means of alternate Numbers." Reference was made to a work by Dr. Hermann Hankel, entitled *Vorlesungen über die Complexen Zahlen und Ihre Functionen* (1867). Mr. Tucker read a paper by the Hon. J. W. Strutt "On the Vibrations of a Gas contained within a Rigid Spherical Case." The problem discussed was referred to in a paper on the "Theory of Rersonance," *Phil. Trans.*, 1871. Its publication seems to be of interest, as it is the only case of the vibration of air within a closed vessel, which has hitherto been solved with complete generality. A result arrived at was that the pitch is about a fourth higher for the sphere than it is for a closed cylindrical pipe, whose length is equal the diameter of the sphere. Mr. A. J. Ellis, F.R.S., stated the following problem which had been proposed to him by Prof. Haldeman, of Pennsylvania, U.S. (who is writing a treatise on English versification): "The number of lines in a rhymed stanza being given, how many variations of rhyme-distribution does it admit of, supposing no line to be left without a rhyme?"

R. Tucker, M.A., *Hon. Sec.*

REVIEWS.

(I) *Reports on the Total Solar Eclipse of August 7th*, 1869. U. S. Naval Observatory, Washington, 1869. 4to. 12 plates, 214 pages.

(II) *Reports on the Total Solar Eclipse of December 29th*, 1870. U. S. Naval Observatory, Washington, 1871. 4to. 2 plates, 132 pages.

These eclipses were observed by Messrs. S. Newcomb, W. Harkness, J. R. Eastman, A. Hall, Professors of Mathematics, U. S. N., and by several experts in Astronomy, among whom we may name, for 1869, Mr. W. S. Gilman, jun., of New York, and for 1870, Capt. G. L. Tupman, R. M. A. of H. M. S. Prince Consort. Photographs were taken in 1869 by Dr. Edward Curtis, U. S. A. In general the observers saw the first eclipse at Des Moines, Iowa, U. S., and the second eclipse at Syracuse, in Sicily.

The latitude of each observing station was determined by sextant and the longitude by telegraph. From the detailed accounts of the operations members of future Astronomical Expeditions may learn what devices are called for in pressing emergencies. We especially admire the ingenious arrangement adopted by Prof. Harkness for recording, on the same fillet of paper and by one pen moved by a single magnet, every fifth second of Maltese mean-time and every second of mean-time at Syracuse.

A telegraphic system more than 1300 miles in length connects Des Moines with Washington. Syracuse is connected *viâ* Malta with Gibraltar by a submarine telegraph proceeding *viâ* Lisbon to Falmouth. Owing to a break-down in the line at Lisbon, Prof. Newcomb, who was at Gibraltar in 1870, could telegraph only to Malta, and was unable to communicate, as had been arranged, with the Royal Observatory, Greenwich.

The observations may be classed according to the instruments employed. (1°) *Telescope*, (2°) *Spectroscope*, (3°) *Polariscope*, (4°) *Sun-Camera*, (5°) *Actinometer, Photometer*, *Barometer*, *Thermometer*.

(1°) *Telescopes* were used to observe the general phenomena and the times of first and last contact, and of the beginning and end of totality. Improving upon a method adopted by him in 1869, Prof. Newcomb furnished his telescope in 1870 with eleven parallel wires crossed at right angles by four others. The intervals between the wires are given by transits of equatorial stars. If then one set of wires be made parallel to the line of cusps of the partially eclipsed sun it is easy to note the time at which the line of cusps when at right angles to any two parallel wires extends exactly from one to the other.

In 1869 totality lasted 2 m. 57·2 s. at Des Moines, and 2 m. 47·5 s. at Sioux, Iowa. In 1870 the duration was 1 m. 51 s. at Gibraltar, and 1 m. 45·5 s. at Syracuse. Amateur observers scattered over America in 1869 were specially charged to notice the duration of totality, in order to determine the path of the moon's shadow on the Earth's surface, by a method identical with that adopted by Halley in 1715 on the occasion of the last total solar eclipse visible in England.

The Corona, as seen by Prof. Eastman at Syracuse, consisted, as in 1869, of three distinct portions, Near the edge of the moon the corona was nearly white like the denser parts of nebulæ. This luminous ring, much obscured by low continuous ranges of red prominences, was about 1′ in height. Next beyond this ring the corona extended for about 6′ with a decidedly radial structure especially outwards. This part was concentric with the sun and circular in outline. The third or outer part of the corona consisted of projections of light, striated, with a radial structure, like the short bands of streamers seen rising from the auroral arch. In each volume will be found two or more coloured drawings of the eclipsed sun and its appendages as seen during totality by different observers.

Prof. Hall reports that when the totality began at Syracuse, the prominences darted quickly into view and the sunlight flashed back, causing an apparent mingling of red and white light. This effect is well represented in Mr. Gilman's picture of the corona seen by him in 1869. Mr. Hall noticed a rift in the corona at the S.W. part of the moon's limb, and observed also that where the clouds were less dense, long streamers seemed to shoot down the corona.

Capt. Tupman, who was engaged at Syracuse in directing the finder of Prof. Harkness' spectroscope, reports that the colour of the prominences was a strong apricot pink, unlike any colour in the solar spectrum. The body of the moon was

considerably illuminated with a greenish-gray tint, like the lumiére cendrée seen at new moon. The moon was not so dark as the sky beyond the corona. Seen with the low power of 10 diameters the corona was made up entirely of fine black lines on a white ground.

Mr. Farrell, at Sioux, Iowa, describes the 1869 corona thus: "It was a silvery-gray crown of light, as if countless fine jets of steam were issuing from behind a dark globe. Near the moon's disc the light seemed almost phosphorescent." Mr. W. S. Gilman, at the same time and place, using a power of 50 with 4-inch aperture, says the corona was composed of an infinitude of fine violet, mauve-coloured white, and yellowish-white rays issuing from behind the moon.

In 1869, Prof. Eastman described the colour of the prominences as carmine, a dark shade near the base changing to bright pink at the summit. Nearest the sun the corona consisted of a nearly continuous band 1′ in height, silvery-white and of uniform density, a mass of nebulous light like the most brilliant parts of the milky-way. From this band rays proceeded outwards, generally radial, the bases of the rays were of the same light as the inner band, but the outer portions had a very faint greenish-violet light. No change could be observed in the colour or position of these rays during totality. A rush of almost tangible darkness appeared at the beginning of totality, and it seemed as if a screen had been withdrawn to present the corona as a back-ground for the better exhibition of the black body of the moon and the coloured prominences.

Baily's beads. Prof. Eastman, in 1870, noticed that the solar crescent finally broke up into four pieces, which vanished together. General Myer and Colonel Winthrop, U. S. A., observing, in 1869, from White Top Mountain in Virginia, 5530 feet above sea-level, report that their guides saw Baily's beads very readily with unaided eyes, exclaiming that the sun was "breaking to pieces." These guides could also distinguish some of the seven visible prominences which were constant in hue and of a deep rose colour. General Myer reports that the straight silvery rays of the corona were without motion and seemed to converge to a centre.

Prof. Harkness, at Des Moines in 1869, saw one prominences at least with the unaided eye. He considers that the existence of rays, streamers, and rifts in the corona cannot be doubted. It was Dr. Wyberd who, in 1652, first started the false notion that the corona exhibits a rotary motion.

(2°) *Spectroscope.* The finder of the spectroscope-telescope was provided with a needle point which could be placed by an assistant on the upper or lower part of a prominence or on any part of the corona. The assistant carefully noted down the situation and appearance of the prominences, and the places on the corona towards which the needle points. Meanwhile the optical image remained on the slit of the spectroscope, and Prof. Harkness compared the lines of its spectrum with the divisions of an illuminated scale.

In 1869 five prominences were examined. Their spectra consisted of *bright* lines identical in situation with *C*, *D*, *F*, and a green line (probably 1474 of Kirchoff), which was the only line common to the corona and the prominences. The spectrum of the corona seen through a slit widened to 0·003 inch consisted of one bright green line on a faint continuous spectrum. According to Lockyer, hydrogen, when comparatively cool, yields a continuous spectrum with one bright line *F*, this may account for the *continuous* part of the coronal spectrum.

The *number* of lines visible on a prominence spectrum depends solely on the part of it observed. If the spectrum is taken from the summit, but few bright lines are obtained. More lines are obtained from lower points, and if the spectrum be taken from near the base of a prominence all the former lines are seen with some newer ones in addition. All the prominences, therefore, possess the same physical constitution, which, however, changes from their bases to their summits.

The corona does not shew the Fraunhofer dark lines on its spectrum, and cannot, therefore, be an effect of reflected sunlight. Prof. Harkness considers the corona to be a gaseous envelope of the sun. He believes the coronal line, green, to be due to the vapour of iron. The disappearance of some of the hydrogen lines, but the continuance of the red hydrogen line *C* up to the very top of the prominences, shews that the temperature of their summits is at least 2500° *C*, which is more than sufficient to vapourise iron. In passing upwards along a prominence, we lose, first, the magnesium lines *b* and one blue hydrogen line *F*, then the green line (iron ?), next the unknown yellow line near *D*, and, finally, the red hydrogen line *C* alone remains at the summit.

In 1870, at Syracuse, the wind was so high as to blow out the light of two lamps successively used to illuminate the scale. No readings for position of lines could be taken. Mr. Harkness can state only that the coronal line is green, and at

about the same place as when he saw it at Des Moines, *i.e.* 1474, Kirchoff. Two other fainter green lines undoubtedly belong to the corona; they were seen to be less refrangible than 1474. Prof. Harkness' conclusion is that "when seen in a clear sky the corona is a purely solar phenomenon, produced by a vast body of self-luminous gas, not improbably incandescent vapour of iron, which envelopes the sun and is thrown out in the same manner as the red prominences."

(3°) *Polariscope.* In 1869 no observations of any value were made. In 1870 Prof. Harkness used an Arago-Polariscope, consisting of a plate of Selenite and a double image prism, fitted to an eye-piece provided with a diaphragm of such diameter that when seen through the polariscope two circular fields appear to touch each other. If polarised light is present these fields are of complementary colours. When the corona is viewed through this instrument the arrangement of the tints will shew whether the polarisation is radial or confined to a single plane. It is inferred from the observations that the sky and the corona were both polarised to the same extent, and it is supposed that the polarisation is due to our atmosphere, and that the light when first emitted from the corona is not polarised at all. Mr. Griffiths, of the English Expedition, observed that the amount of polarisation of the light of the sky increased from the moon's centre outwards. Such light during an eclipse could be derived from the corona only, but the increase can only be accounted for by supposing the polarisation to be effected in our atmosphere.

Prof. Eastman used a Savart-Polariscope, consisting of a plate of quartz cut obliquely to the axis, and a plate of tourmaline, giving Savart's bands when polarised light is present. With this instrument he saw alternate light and dark bands at the centre of the moon's dark surface. These bands were unchanged in distinctness or tint for a complete revolution of the Savart. Over a belt of sky beyond the corona exactly the same light and dark bands were seen, but they seemed most intense wherever there was cloud. On the densest parts of the corona itself the results were the same, except that the intensity of the bands was a maximum or minimum when they were tangential or radial to the moon's limb. On the corona he once saw a faint tinge of green in the bands, but could not see it again. On the bright edges of clouds he saw faint but decided traces of red tints in the bands.

(4°) *Sun-Camera.* In 1870 no provision for photography was made. In 1869 Dr. Edward Curtis, U.S.A., succeeded in taking two photographs, although the image of the corona projected by the finder on a screen was barely visible through the prevailing haze. These photographs required exposures of 66 s. and 45 s., and there was no time left for another attempt during totality. In every photograph of the partial phase a bright line is to be seen along the projection of the moon's dark limb on the sun's disc. This line is not an effect of refraction nor of diffraction. It has been shewn by Prof. H. Morton's experiments to be a chemical effect produced in developing the photographs. At Syracuse, Prof. Harkness and Capt. Tupman looked attentively, but in vain, for this supposed bright line. Once or twice only could they fancy that they saw any trace of it, and even then they consider what they noticed to be an effect due to contrast between the dark moon and the bright sun.

(5°) *Actinometer, Photometer, Barometer, Thermometer.* The observations require no special comment.

The United States Government may justly take pride in the achievements of its astronomers specially commissioned to note the phenomena of two eclipses in which totality was of but too brief duration. Solar science is indeed largely indebted to these observers for the care, skill, and fidelity displayed in their reports, and in addition for the prompt issue of them in a collective form.

A. Freeman.

END OF VOL. I.

PRINTED BY W. METCALFE AND SONS, GREEN STREET, CAMBRIDGE.

Fig. 1

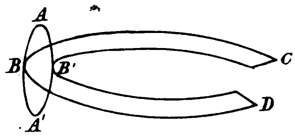


Fig. 2

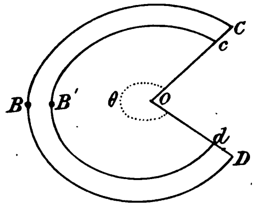
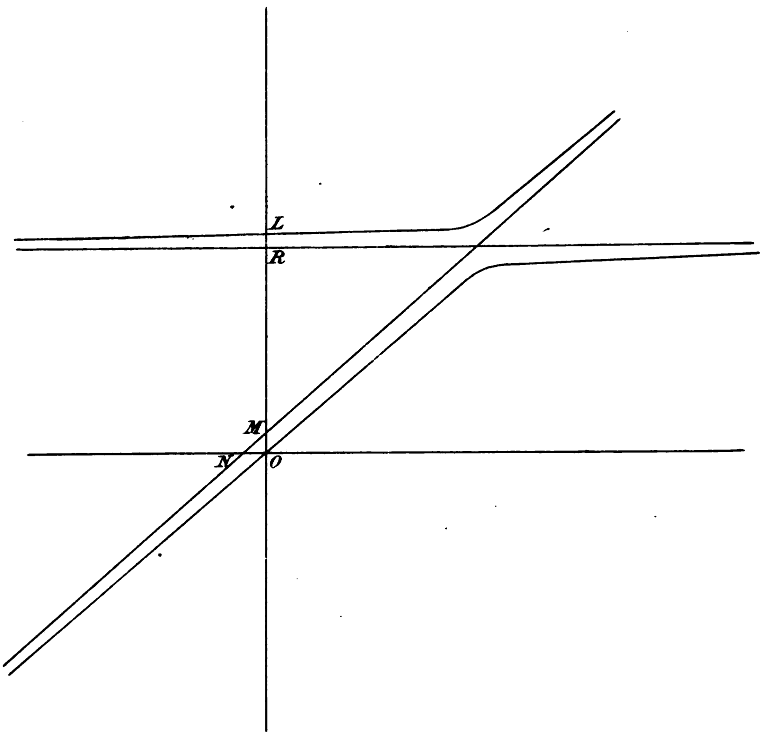


Fig. 3.



W. Metcalfe

Fig. 9

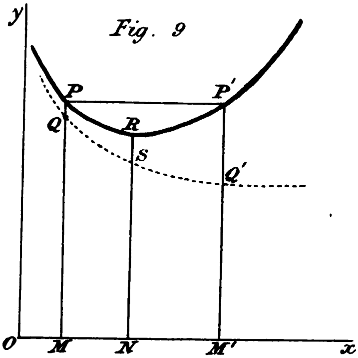


Fig. 11

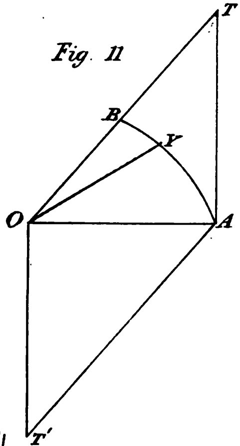


Fig. 10

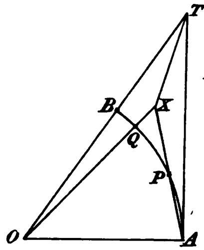
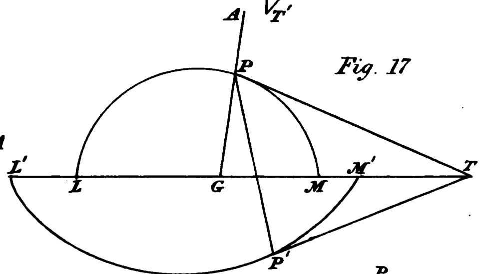


Fig. 17



ig. 14

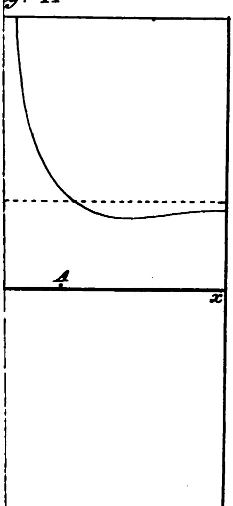


Fig. 18

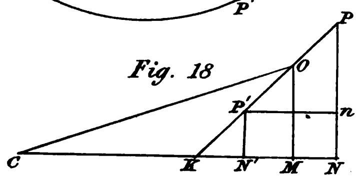


Fig. 19

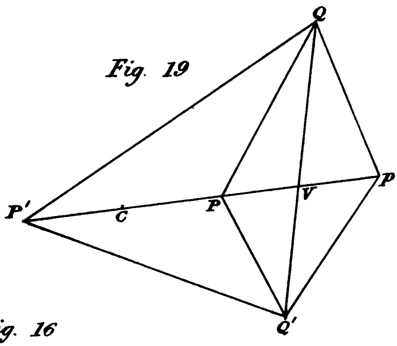


Fig. 16

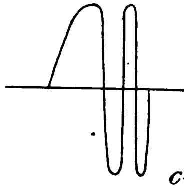
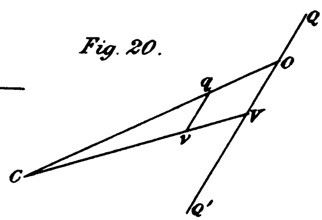
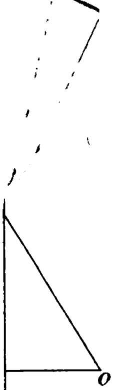


Fig. 20.





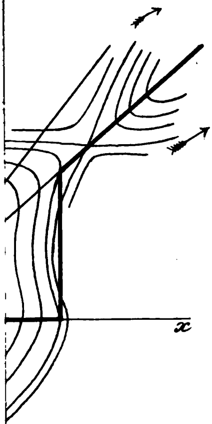


Fig. 25

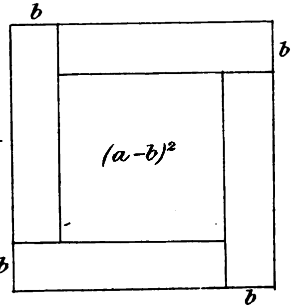
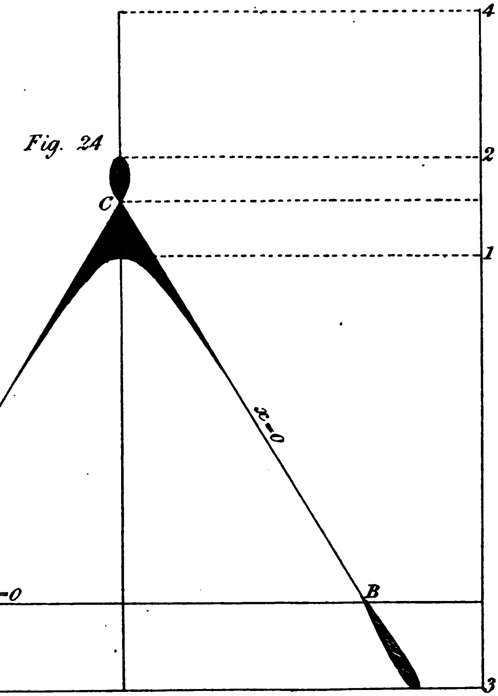


Fig. 24



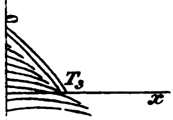
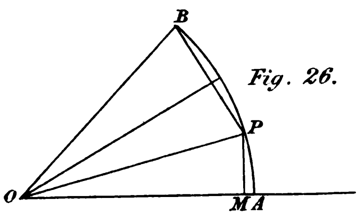


Fig. 26.



W. Metcalfe & Sons, Litho.

9